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Inverse Problems for Ergodicity of Markov Chains

by

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in partial fulfillment for
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马氏链遍历性反问题

摘　要

本文主要讨论离散时间及连续时间马氏链几种遍历性质的反问题, 试图给出易用的判别准则.

第一章介绍了相关研究背景, 回顾了已有的马氏链遍历性质实用判别准则. 同时, 我们在这一章中叙述了本文的主要工具 — 最小非负解理论 — 的相关结果.

第二章叙述了我们得到的遍历性反问题的主要结果. 对于连续时间参数的马氏链（即 Q 过程）, 我们给出了四种遍历性质（包括强遍历性、指数遍历性、代数遍历性以及遍历性）反问题的较为易用的判别准则. 平行地, 我们给出了离散时间马氏链的强遍历性、代数遍历性和遍历性反问题的判别准则.

第三章给出了反问题判别准则的证明. 为证条件的充分性, 我们对马氏链回访时的矩（包括多项式阶矩和指数阶矩）给出下端控制. 另一方面, 我们应用有限逼近方法说明所给条件的必要性.

第四章叙述了一些应用. 首先在第 1 节中, 利用反问题判别准则, 我们得到了单生过程遍历性和强遍历性显式判别准则（必要性部分）的一个新证明. 在第 2 节中, 我们利用反问题判别准则及耦合等工具处理了一小类特殊的单生过程, 得到了过程的一些遍历性条件. 在第 3 节中, 我们将反问题判别准则应用到有限维 Brussel 模型, 给出了该模型非强遍历性的新证明.

关键词:　马氏链, Q 过程, 遍历性, 反问题

Inverse Problems for Ergodicity of Markov Chains

ABSTRACT

In this thesis, we study inverse problems for ergodicity of Markov Chains (both time-continuous and time-discrete) and intend to give practical criteria.

Chapter 1 introduces research background and recalls practical criteria for ergodicity of Markov Chains. Meanwhile, we present some results in minimal solution theory, which is our main tool.

Chapter 2 includes our main results. For Continuous-Time Markov Chains, we obtain inverse problem criteria for strong ergodicity, exponential ergodicity, algebraic ergodicity and (ordinary) ergodicity. Similarly, we study inverse problems for strong ergodicity, algebraic ergodicity and (ordinary) ergodicity in time-discrete case.

Chapter 3 gives proofs of our main results. To prove sufficiency, we estimate lower control for polynomial moments and exponential moments of return time. On another hand, applying finite approximation method, we prove necessity of our criteria.

Some applications are included in Chapter 4. Using our inverse problem criteria, in Section 1, we give new proofs of explicit criteria for both ergodicity and strong ergodicity of single birth processes (necessity part). In Section 2, we study a special class of single birth processes and obtain some of its ergodic conditions. In Section 3, inverse problem criteria are applied to finite-dimensional Brussel's Model. And a new proof of non-strong ergodicity of finite-dimensional Brussel's model is presented.

KEY WORDS: Markov Chain, Q-process, Ergodicity, Inverse Problems

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Chapter 0

综　述

马氏链的遍历性理论是马氏链研究的重要组成部分, 在理论与应用中有广泛价值, 受到学者们的关注. Q 过程（连续时间马氏链）的经典遍历性质有以下四种：令 $Q=(q_{ij}:i,j\in E)$ 为可数状态空间 E 上的 Q 矩阵, $P(t)=(p_{ij}(t))$ 为 Q 过程, $\pi=(\pi_i)_{i\in E}$ 为过程的平稳分布, 我们称该 Q 过程为

(1) 遍历的, 若对任意的 $i\in E$, $\lim\limits_{t\to\infty}\| p_{i\cdot}(t)-\pi \|_{\mathrm{Var}}=0$;

(2)（代数遍历性）ℓ 遍历的 ($\ell\geqslant 1$), 若对任意的状态 i 和 j, $|\ p_{ij}(t)-\pi_j\ |=O\left(t^{-(\ell-1)}\right)$ (当 $t\to\infty$);

(3) 指数遍历的, 若对任意的状态 i 和 j, 存在 $\beta>0$ 使得 $|p_{ij}(t)-\pi_j|=O\left(\mathrm{e}^{-\beta t}\right)$ (当 $t\to\infty$);

(4) 强遍历的, 若 $\lim\limits_{t\to\infty}\sup\limits_{i\in E}\| p_{i\cdot}(t)-\pi \|_{\mathrm{Var}}=0$.

同时上述四种遍历性有如下概率刻画（参看文献 [1,4]）：记 η_1 为 Q 过程首次跳的时间, H 为 E 的非空有限子集, $\sigma_H\hat{=}\inf\{t\geqslant\eta_1:X_t\in H\}$, 则 Q 过程

(1) 遍历, 当且仅当 $\max\limits_{i\in H}\mathbb{E}_i\sigma_H<\infty$;

(2) ℓ 遍历 ($\ell\geqslant 1$), 当且仅当 $\max\limits_{i\in H}\mathbb{E}_i\sigma_H^{\ell}<\infty$;

(3) 指数遍历, 当且仅当对某个满足 $0<\lambda<q_i$ ($\forall\ i\in E$) 的 λ, 成立着 $\max\limits_{i\in H}\mathbb{E}_i\mathrm{e}^{\lambda\sigma_H}<\infty$;

(4) 强遍历, 当且仅当 $\sup\limits_{i\notin H}\mathbb{E}_i\sigma_H<\infty$.

注意到上面的第 (2) 条, 为叙述方便起见, 当过程常返时我们也称其为 0 遍历的. 后来得到的马氏链遍历性易用判别准则使相关研究进一步完善. 作为最小非负解理论的应用, 侯振挺教授于 1978 年给出了如下 Q 过程遍历性判别准则：

定理 1 ([17, 定理 9.4.1]). 令 Q 为不可约正则的 Q 矩阵, H 为状态空间 E 的非空有限子集. 那么 Q 过程遍历的充分必要条件是下面不等式

$$\begin{cases} \sum_{j\in E} q_{ij}y_j \leqslant -1, & i \notin H, \\ \sum_{i\in H}\sum_{j\neq i} q_{ij}y_j < \infty \end{cases}$$

存在有限的非负解.

1981 年, Tweedie 在文献 [11] 中进一步给出了 Q 过程指数遍历和强遍历的实用判别准则:

定理 2 ([11, Theorem 2]). 令 Q 为不可约正则的 Q 矩阵, H 为状态空间 E 的非空有限子集. 那么 Q 过程指数遍历的充分必要条件是, 对于某个满足 $0 < \lambda < q_i$ $(\forall i \in E)$ 的 λ, 下面不等式

$$\begin{cases} \sum_{j\in E} q_{ij}y_j \leqslant -\lambda y_i - 1, & i \notin H, \\ \sum_{i\in H}\sum_{j\neq i} q_{ij}y_j < \infty \end{cases}$$

存在有限的非负解.

定理 3 ([11, Theorem 2]). 令 Q 为不可约正则的 Q 矩阵, H 为状态空间 E 的非空有限子集. 那么 Q 过程强遍历的充分必要条件是下面不等式

$$\sum_{j\in E} q_{ij}y_j \leqslant -1, \qquad i \notin H \tag{1}$$

存在有界解.

2004 年, 毛永华教授在文献 [10] 中给出了 Q 过程代数遍历性的实用判别准则:

定理 4 ([10, Theorem 1.5]). 令 Q 为不可约正则的 Q 矩阵, H 为状态空间 E 的非空有限子集, ℓ 为一非负整数. 设 Q 过程 ℓ 遍历, 则 Q 过程 $(\ell+1)$ 遍历的充分必要条件是如下不等式

$$\begin{cases} \sum_{j\in E} q_{ij}y_j \leqslant -(\ell+1)\,\mathbb{E}_i\sigma_H^{\ell}, & i \notin H, \\ \sum_{i\in H}\sum_{j\neq i} q_{ij}y_j < \infty \end{cases}$$

存在有限的非负解.

应用以上判别准则, 我们可以从某不等式解的存在性推知过程的遍历性质. 然而, 反过来, 比如要用定理 3 判断某 Q 过程不是强遍历的, 就不很方便. 事实上, 欲应用定理 3 得到过程的非强遍历性, 我们必须说明不等式 Eq. (1) 不存在有界解, 而这难于操作. 实际上, 陈木法教授曾在 1985 年前后给出了如下的“一对” Q 过程惟一性和非惟一性的实用判别准则:

定理 5 ([4, Theorem 2.25]). *令 Q 为 E 上全稳定保守的 Q 矩阵, $\{E_n\}_{n=1}^{\infty}$ 为 E 的一列递升子集, φ 为 E 上的非负函数. 若*

(1) $\sup_{i\in E_n} q_i < \infty,\ \lim_{n\to\infty} \inf_{i\notin E_n} \varphi_i = \infty,$

(2) *存在常数 $c \in \mathbb{R}$, 使得*

$$Q\varphi \leqslant c\varphi,$$

那么 Q 过程惟一.

定理 6 ([4, Theorem 2.27]). *若对于某个 $c > 0$, 不等式*

$$Q\varphi \geqslant c\varphi$$

存在有界解 φ 满足 $\sup_{i\in E} \varphi_i > 0$, 则 Q 过程非惟一.

这两个判别准则“双剑合璧”, 可以方便地处理 Q 过程惟一性问题. 同时, 我们注意到 Kim 等曾在文献 [7,8] 中给出了 Q 过程非遍历的两个充分条件, 如

定理 7 ([7, Theorem 1]). *令 $Q = (q_{ij})$ 为 E 上不可约正则的 Q 矩阵. 若有 E 上的非常值的实值函数 f 满足*

$$\sup_{i\in E} \sum_{j\in E} q_{ij} (f_i - f_j)^+ < \infty,$$
$$\sum_{j\in E} q_{ij} f_j \geqslant 0, \qquad i \in E,$$

则 Q 过程非遍历.

本篇学位论文的目标即为讨论遍历性质反问题的判别准则, 即：能否通过某不等式及其（满足某些条件的）解的存在性推知过程非遍历（或者非代数遍历、非指数遍历以及非强遍历)?

若将目光限于单生过程, 几种遍历性的显式可计算的判别条件已经得到. 严士健教授和陈木法教授于 1986 年给出了单生过程遍历性的显式判别准则. 2001 年, 张余辉教授给出了单生过程强遍历性显式判别准则. 2004 年, 毛永华教授和张余辉教授给出了单生过程指数遍历性的一个显式可计算的充分条件. 进一步地, 吴波和张余辉教授在文献 [12] 中利用与单生过程比较的办法给出了多维 Q 过程遍历性质的一些必要条件. 这些具体结果为我们遍历性反问题的研究提供了宝贵的“参照物”.

为研究遍历性反问题, 我们首先回顾文献 [4] 中给出的定理 3 的证明. 事实上, $(\mathbb{E}_i\sigma_H)_{i\notin H}$ 是方程

$$x_i = \sum_{\substack{j\notin H\\ j\neq i}} \frac{q_{ij}}{q_i} x_j + \frac{1}{q_i}, \qquad i \notin H$$

的最小非负解（参见 [4, 定理 4.48]）. 于是由最小非负解的比较原理, 如下不等式

$$y_i \geqslant \sum_{\substack{j\notin H\\ j\neq i}} \frac{q_{ij}}{q_i} y_j + \frac{1}{q_i}, \qquad i \notin H \tag{2}$$

的任意非负解 $(y_i)_{i\notin H}$ 满足

$$\mathbb{E}_i\sigma_H \leqslant y_i, \qquad i \notin H.$$

从而要使 $(\mathbb{E}_i\sigma_H)_{i\notin H}$ 有界, 只要不等式 Eq. (2) 存在一个非负有界解, 稍作整理即得定理 3.

从定理 3 的证明可知, 关键点是（利用最小非负解的比较原理）找到了 $(\mathbb{E}_i\sigma_H)_{i\notin H}$ 的一个上端控制. 在考虑强遍历性反问题时, 我们希望可以（对称地）找到 $(\mathbb{E}_i\sigma_H)_{i\notin H}$ 的一个下端控制. 此时最小解和最大解理论均已不再适用, 但我们仍得到

引理 8. 若 Q 过程遍历, 则下面不等式

$$y_i \leqslant \sum_{\substack{j\notin H\\ j\neq i}} \frac{q_{ij}}{q_i} y_j + \frac{1}{q_i}, \qquad i \notin H$$

的任意一个上有界解（未必非负） $(y_i)_{i\notin H}$ 满足

$$\mathbb{E}_i\sigma_H \geqslant y_i, \qquad i \notin H.$$

该引理提供了 $(\mathbb{E}_i\sigma_H)_{i\notin H}$ 的一个下端控制. 留意, 上述引理中的“上有界”条件不可去掉, 请参见 Remark 21 给出的说明. 由该引理, 立得如下定理的充分性：

定理 9. 令 Q 为不可约正则的 Q 矩阵, H 为状态空间 E 的非空有限子集. 则 Q 过程非强遍历的充分必要条件是存在 $E\setminus H$ 上的函数列 $\left\{y^{(n)}\right\}_{n=1}^{\infty}$, 其中对每个 $n\geqslant 1$, $y^{(n)} = \left(y_i^{(n)}\right)_{i\notin H}$, 且 $\{y^{(n)}\}_{n=1}^{\infty}$ 满足如下条件：

(1) 对每个 $n \geqslant 1$, $\sup\limits_{i\notin H} y_i^{(n)} < \infty$;

(2) 对每个 $n \geqslant 1$, $\left(y_i^{(n)}\right)_{i\notin H}$ 满足

$$y_i \leqslant \sum_{j\notin H} \Pi_{ij} y_j + \frac{1}{q_i}, \qquad i \notin H;$$

(3) $\overline{\lim\limits_{n\to\infty}}\sup\limits_{i\notin H} y_i^{(n)}=\infty.$

上述定理中的 $\Pi=(\Pi_{ij}:i,j\in E)$ 为 Q 过程的嵌入链, 即

$$\Pi_{ij}\widehat{=}(1-\delta_{ij})\frac{q_{ij}}{q_i},\qquad i,j\in E.$$

我们再来简述定理 9 必要性的证明. 不失一般性, 设 $E=\{0,1,2,\cdots\}$ 及 $H=\{0\}$, 记方程

$$x_i=\sum_{\substack{1\leqslant j\leqslant n\\ j\neq i}}\frac{q_{ij}}{q_i}x_j+\frac{1}{q_i},\qquad 1\leqslant i\leqslant n$$

的最小非负解为

$$x^{(n)}=\left(x_i^{(n)},\ 1\leqslant i\leqslant n\right),$$

则 $x^{(n)}$ 存在有限（参看 Lemma 22）, 且

$$\mathbb{E}_i\sigma_0=\lim_{n\to\infty}\uparrow x_i^{(n)},\qquad i\geqslant 1.$$

由此, 即可证得定理 9 的必要性.

使用类似的“下端控制”和“有限逼近”手法, 可以得到代数遍历性和指数遍历性反问题的判别准则, 命题的证明在技术上复杂了许多.

定理 10. 令 Q 为不可约正则的 Q 矩阵, H 为状态空间 E 的非空有限子集, ℓ 为一取定的非负整数. 设 Q 过程为 ℓ 遍历的, 则 Q 过程不 $(\ell+1)$ 遍历的充分必要条件是存在 E 上的函数列 $\{y^{(n)}\}_{n=1}^{\infty}$, 其中对每个 $n\geqslant 1$, $y^{(n)}=\left(y_i^{(n)}\right)_{i\in E}$, 且 $\{y^{(n)}\}_{n=1}^{\infty}$ 满足如下条件:

(1) 对每个 $n\geqslant 1$, $\sup\limits_{i\in E} y_i^{(n)}<\infty$;

(2) 对每个 $n\geqslant 1$, $\left(y_i^{(n)}\right)_{i\in E}$ 满足

$$y_i\leqslant\sum_{j\notin H}\Pi_{ij}y_j+\frac{(\ell+1)}{q_i}\mathbb{E}_i\sigma_H^{\ell},\qquad i\in E;$$

(3) $\overline{\lim\limits_{n\to\infty}}\max\limits_{i\in H} y_i^{(n)}=\infty.$

定理 11. 令 Q 为不可约正则的 Q 矩阵且 $\inf\limits_{i\in E} q_i>0$, H 为状态空间 E 的非空有限子集. 则 Q 过程非指数遍历的充分必要条件是存在正数列 $\{\lambda_n\}_{n=1}^{\infty}$ 和 E 上的函数列 $\{y^{(n)}\}_{n=1}^{\infty}$ 满足如下条件:

(1) $\lim\limits_{n\to\infty}\lambda_n=0$;

(2) 对每个 $n \geqslant 1$, $\left(y_i^{(n)}\right)_{i \in E}$ 为有限支撑的且满足

$$y_i^{(n)} \leqslant \frac{q_i}{q_i - \lambda_n} \sum_{j \notin H} \Pi_{ij} y_j^{(n)} + \frac{1}{q_i - \lambda_n}, \qquad i \in E;$$

(3) $\varlimsup_{n \to \infty} \max_{i \in H} y_i^{(n)} = \infty$.

值得一提的是, 在处理指数遍历性反问题时, 我们所做的下端控制基于如下的最小非负解结论. 该命题在一定条件下给出了最小解的一个下端控制.

命题 12. 令 f^* 为 $f = Af + g$ $(x \in E)$ 的最小非负解, 非负函数 $\widetilde{f}$ 满足

$$\widetilde{f} \leqslant A\widetilde{f} + g, \qquad x \in E.$$

若对于某非负实数 p 成立 $\widetilde{f} \leqslant pf^*$, 那么 $\widetilde{f} \leqslant f^*$.

同样的“下端控制”和“有限逼近”手法也可以用来处理离散时间马氏链的遍历性反问题. 我们得到了如下遍历性反问题判别准则.

定理 13. 令 $P = (P_{ij})$ 为不可约非周期的转移概率矩阵, H 为状态空间 E 的非空有限子集, ℓ 为一取定的非负整数. 设马氏链为 ℓ 遍历的, 则链不 $(\ell+1)$ 遍历的充分必要条件是存在 E 上的函数列 $\left\{y^{(n)}\right\}_{n=1}^{\infty}$, 其中对每个 $n \geqslant 1$, $y^{(n)} = \left(y_i^{(n)}\right)_{i \in E}$, 且 $\left\{y^{(n)}\right\}_{n=1}^{\infty}$ 满足如下条件:

(1) 对每个 $n \geqslant 1$, $\sup_{i \in E} y_i^{(n)} < \infty$;

(2) 对每个 $n \geqslant 1$, $\left(y_i^{(n)}\right)_{i \in E}$ 满足

$$y_i \leqslant \sum_{j \notin H} P_{ij} x_j + \mathbb{E}_i \sigma_H^{\ell}, \qquad i \in E;$$

(3) $\varlimsup_{n \to \infty} \max_{i \in H} y_i^{(n)} = \infty$.

定理 14. 令 $P = (P_{ij})$ 为不可约非周期的转移概率矩阵, H 为状态空间 E 的非空有限子集. 则马氏链非强遍历的充分必要条件是存在 $E \setminus H$ 上的函数列 $\left\{y^{(n)}\right\}_{n=1}^{\infty}$, 其中对每个 $n \geqslant 1$, $y^{(n)} = \left(y_i^{(n)}\right)_{i \notin H}$, 且 $\left\{y^{(n)}\right\}_{n=1}^{\infty}$ 满足如下条件:

(1) 对每个 $n \geqslant 1$, $\sup_{i \notin H} y_i^{(n)} < \infty$;

(2) 对每个 $n \geqslant 1$, $\left(y_i^{(n)}\right)_{i \notin H}$ 满足

$$y_i \leqslant \sum_{j \notin H} P_{ij} y_j + 1, \qquad i \notin H;$$

(3) $\overline{\lim\limits_{n\to\infty}} \sup\limits_{i\notin H} y_i^{(n)} = \infty$.

在本文的最后一章, 我们的遍历性反问题判别准则应用到了若干具体例子上, 得到一些有趣的结论. 例如

命题 15. 考虑 $E = \{0, 1, 2, \cdots\}$ 上的保守单生 Q 矩阵 $Q = (q_{ij})$:

$$q_{ij} = \begin{cases} i+1, & i \geqslant 0,\ j = i+1, \\ \alpha_i > 0, & i \geqslant 1,\ j = 0, \\ 0, & \text{其他 } i \neq j. \end{cases}$$

若 $\lim\limits_{i\to\infty} \alpha_i = 0$, 则 Q 过程非指数遍历.

在第 4 章第 2 节中, 我们给出了该命题基于定理 9 和定理 11 的两种证明.

Chapter 1

Introduction

As an application of his study in minimal non-negative solution theory, Hou gave a practical criterion for ergodicity of Continuous-Time Markov Chains in [17]. Then in 1981, Tweedie [11] studied criteria for three classical types of ergodicity: (ordinary) ergodicity, exponential ergodicity and strong ergodicity. Mao [9, 10] and Chen and Wang [5] studied criteria for algebraic ergodicity. According to their practical criteria, a solution to some inequality group implies, for example, strong ergodicity of a Markov Chain. However, using their criteria, one may find it not that easy to make sure a Q-process, say, not being strongly ergodic. In fact, Tweedie's strong ergodicity criterion reads as follows:

Tweedie's Strong Ergodicity Criterion. *Let Q be an irreducible regular Q-matrix and H a non-empty finite subset of E. Then the Q-process is strongly ergodic iff there exists a bounded solution $(y_i)_{i\in E}$ to inequality*

$$\sum_{j\in E} q_{ij}y_j \leqslant -1, \qquad i \notin H.$$

If we are proving a Q-process is not strongly ergodic using this criterion, we have to show that any solution to the above inequality is unbounded. Neverthless, this is not so exercisable. We intend to complement ergodicity criteria in this thesis. For instance, can we assert non-strong ergodicity of a Q-process from some inequality and one of its solutions?

Our work is based on some shining examples. It is worthy to mention that Chen [2] gives a practical criterion for uniqueness of Q-processes, which is actually an illuminating model and a precurssor for this thesis. Explicit criteria for ergodicity and strong ergodicity of single birth processes (obtained by Yan and Chen [13] and Zhang [14], respectively) serve as concrete examples and catalyst for this thesis.

Since we are dealing with ergodic properties, we assume processes considered are all recurrent without loss of generality. And we will consider not only continuous-time but also discrete-time Markov Chains.

Let's begin with minimal solution theory, which is a powerful tool in the study of Markov Chains.

1 Minimal Solution Theory Preparations

References [4, Chapter 2] or [17, Chapter 2] give a thorough introduction to minimal solution theory. Most propositions in this section are taken from [4, 17], except for our original Lemma 7.

Let E be an arbitrary non-empty set. Denote by $\mathscr{H}$ a set of mappings from E to $\overline{\mathbb{R}}_+ \widehat{=} [0, +\infty]$: $\mathscr{H}$ contains the constant 1 and is closed under non-negative linear combination and monotone increasing limit, where the order relation "$\geqslant$" in $\mathscr{H}$ is defined pointwisely. Then $\mathscr{H}$ is a convex cone. We say A: $\mathscr{H} \to \mathscr{H}$ is a cone mapping if $A0 = 0$ and

$$A\left(c_1 f_1 + c_2 f_2\right) = c_1 A f_1 + c_2 A f_2, \qquad \text{for all } c_1, c_2 \geqslant 0 \text{ and } f_1, f_2 \in \mathscr{H}.$$

Denote by $\mathscr{A}$ the set of all such mappings which also satisfy the following hypothesis:

$$\mathscr{H} \ni f_n \uparrow f \text{ implies } Af_n \uparrow Af.$$

Definition 1. Given $A \in \mathscr{A}$ and $g \in \mathscr{H}$. We say f^* is a minimal non-negative solution (abbrev. minimal solution) to equation

$$f = Af + g, \qquad x \in E, \tag{3}$$

if f^* satisfies Eq. (3) and for any solution $\widetilde{f} \in \mathscr{H}$ to Eq. (3), we have

$$\widetilde{f} \geqslant f^*, \qquad x \in E.$$

Theorem 2 ([4, Theorem 2.2])**.** *The minimal solution to Eq.* (3) *always exists uniquely.*

Definition 3. Let $A, \widetilde{A} \in \mathscr{A}$ and $g, \widetilde{g} \in \mathscr{H}$ satisfy

$$\widetilde{A} \geqslant A, \quad \widetilde{g} \geqslant g.$$

Then we call

$$\widetilde{f} \geqslant \widetilde{A}\widetilde{f} + \widetilde{g}, \qquad x \in E \tag{4}$$

a controlling equation of Eq. (3).

Theorem 4 ([4, Theorem 2.6], Comparison Principle)**.** *Let f^* be the minimal solution to Eq.* (3)*. Then for any solution $\widetilde{f}$ to Eq.* (4)*, we have $\widetilde{f} \geqslant f^*$.*

By Theorem 2, we may define a map

$$m_A : \mathscr{H} \to \mathscr{H},$$
$$g \mapsto m_A g,$$

where $m_A g$ denotes the minimal solution to Eq. (3).

Theorem 5 ([4, Theorem 2.7]). *m_A is a cone mapping. For $\{A_n\} \subset \mathscr{A}$, $A_n \uparrow A$ and $\{g_n\} \subset \mathscr{H}$, $g_n \uparrow g$, we have $A \in \mathscr{A}, g \in \mathscr{H}$ and $m_{A_n} g_n \uparrow m_A g$.*

Theorem 6 ([4, Theorem 2.10]). *Given an arbitrary non-negative $\widetilde{f}^{(0)}$ satisfying $0 \leqslant \widetilde{f}^{(0)} \leqslant p f^*$ for some non-negative number p, set*

$$\widetilde{f}^{(n+1)} = A\widetilde{f}^{(n)} + g, \qquad n \geqslant 0.$$

Then we have $\widetilde{f}^{(n)} \to f^*$ $(n \to \infty)$.

Lemma 7. *Let f^* be the minimal solution to Eq. (3) and $\widetilde{f}$ be a non-negative function satisfying*

$$\widetilde{f} \leqslant A\widetilde{f} + g, \qquad x \in E. \tag{5}$$

If $\widetilde{f} \leqslant p f^*$ for some non-negative number p, then $\widetilde{f} \leqslant f^*$.

Proof. Assume $p > 1$ without loss of generality. Define

$$\widetilde{f}^{(0)} = \widetilde{f},$$
$$\widetilde{f}^{(n+1)} = A\widetilde{f}^{(n)} + g, \qquad n \geqslant 0.$$

We claim

$$\widetilde{f}^{(n)} \uparrow f^*, \qquad \text{as } n \to \infty.$$

In fact, by Theorem 6, we have

$$\widetilde{f}^{(n)} \to f^*, \qquad \text{as } n \to \infty.$$

So we need only prove the monotonicity. According to Eq. (5),

$$\widetilde{f}^{(0)} \leqslant A\widetilde{f}^{(0)} + g = \widetilde{f}^{(1)}.$$

Now if $\widetilde{f}^{(n)} \leqslant \widetilde{f}^{(n+1)}$ for some $n \geqslant 0$, then

$$\widetilde{f}^{(n+1)} = A\widetilde{f}^{(n)} + g \leqslant A\widetilde{f}^{(n+1)} + g = \widetilde{f}^{(n+2)}.$$

So the monotonicity holds by induction. It follows immediately that

$$\widetilde{f} = \widetilde{f}^{(0)} \leqslant f^*.$$

Lemma 7 is proved. □

2 Preliminaries for Continuous-Time Markov Chains

Consider an irreducible regular Q-matrix $Q = (q_{ij} : i, j \in E)$ on a countable state space E with transition probability matrix $P(t) = (p_{ij}(t))$. We say the Q-process is

(1) ergodic, if for each i, $\| p_{i\cdot}(t) - \pi \|_{\mathrm{Var}} \widehat{=} \sum_{j\in E} |p_{ij}(t) - \pi_j| \to 0$, as $t \to \infty$;

(2) (algebraic ergodicity) ℓ-ergodic for some integer $\ell \geqslant 1$, if for each $i, j \in E$, $| p_{ij}(t) - \pi_j | = O\left(t^{-(\ell-1)}\right)$ as $t \to \infty$;

(3) exponentially ergodic, if for each $i, j \in E$, $|p_{ij}(t) - \pi_j| = O\left(\mathrm{e}^{-\beta t}\right)$ as $t \to \infty$ for some $\beta > 0$;

(4) strongly ergodic, if $\lim\limits_{t\to\infty} \sup\limits_{i\in E} \| p_{i\cdot}(t) - \pi \|_{\mathrm{Var}} = 0$.

Note that we say a Q-process is 0-ergodic when it is recurrent for ease of terminology. Also, we say a Q-process is 1-ergodic if it is ergodic.

Set

$$\sigma_H \widehat{=} \inf\{t \geqslant \eta_1 : X_t \in H\}, \qquad H \subset E,$$

where $(X_t)_{t\geqslant 0}$ is the Q-process and η_1 is the first jump time. There are probabilistic descriptions of above notions. A Q-process is

(1) ergodic if and only if (abbr. iff) $\max\limits_{i\in H} \mathbb{E}_i \sigma_H$ is finite for some (equivalently, for any) non-empty finite subset H of E;

(2) (algebraic ergodicity) ℓ-ergodic for some integer $\ell \geqslant 0$ iff $\max\limits_{i\in H} \mathbb{E}_i \sigma_H^\ell$ is finite for some (equivalently, for any) non-empty finite subset H of E;

(3) exponentially ergodic iff $\max\limits_{i\in H} \mathbb{E}_i \mathrm{e}^{\lambda\sigma_H}$ is finite for some λ (with $0 < \lambda < q_i, \forall i \in E$) and some (equivalently, for any) non-empty finite subset H of E;

(4) strongly ergodic iff $(\mathbb{E}_i \sigma_H)_{i\notin H}$ is bounded for some (equivalently, for any) non-empty finite subset H of E.

The following minimal solution assertions are well-known.

Theorem 8 ([10, Theorem 3.1]). *For any $\ell \geqslant 1$, the moments of return times $\mathbb{E}_i\sigma_H, \mathbb{E}_i\sigma_H^2, \cdots, \mathbb{E}_i\sigma_H^\ell$ are inductively the minimal solution to the following ℓ-family of systems for $0 \leqslant n \leqslant \ell - 1$,*

$$x_i^{(n+1)} = \sum_{j\notin H} \Pi_{ij} x_j^{(n+1)} + \frac{(n+1)}{q_i} x_i^{(n)}, \qquad i \in E,$$

where $x_i^{(0)} = 1$ $(i \in E)$ *and* $\Pi = (\Pi_{ij} : i, j \in E)$ *is the embedding chain of the* Q*-process:*

$$\Pi_{ij} \widehat{=} (1 - \delta_{ij}) \frac{q_{ij}}{q_i}, \qquad i, j \in E.$$

When $\ell = 1$, Theorem 8 gives

Theorem 9. $(\mathbb{E}_i \sigma_H)_{i \in E}$ *is the minimal solution to*

$$x_i = \sum_{j \notin H} \Pi_{ij} x_j + \frac{1}{q_i}, \qquad i \in E.$$

Theorem 10 ([4, Theorem 4.48])**.** *For a non-empty finite subset* H *of* E *and* $\lambda > 0$ *with* $\lambda < q_i$ *for all* $i \in E$*, set* $e_{iH}(\lambda) = \frac{1}{\lambda}\left(\mathbb{E}_i \mathrm{e}^{\lambda \sigma_H} - 1\right) = \int_0^\infty \mathrm{e}^{\lambda t} \mathbb{P}_i(\sigma_H > t)\, \mathrm{d}t$ *for each* $i \in E$ *(cf.* [4, Page 148, Equivalence of Theorems 4.45 and 4.44]*). Then* $(e_{iH}(\lambda))_{i \in E}$ *is the minimal solution to*

$$x_i = \frac{q_i}{q_i - \lambda} \sum_{j \notin H} \Pi_{ij} x_j + \frac{1}{q_i - \lambda}, \qquad i \in E.$$

Now we state Hou's, Tweedie's and Mao's criteria.

Theorem 11 ([17, Theorem 9.4.1])**.** *Let* Q *be an irreducible regular* Q*-matrix and* H *a non-empty finite subset of* E*. Then the* Q*-process is ergodic iff there exists a finite non-negative solution* $(y_i)_{i \in E}$ *to the inequality*

$$\begin{cases} \displaystyle\sum_{j \in E} q_{ij} y_j \leqslant -1, & i \notin H, \\ \displaystyle\sum_{i \in H} \sum_{j \neq i} q_{ij} y_j < \infty. \end{cases}$$

Theorem 12 ([10, Theorem 1.5])**.** *Let* Q *be an irreducible regular* Q*-matrix and* H *a non-empty finite subset of* E*. Then the* Q*-process is* ℓ*-ergodic* $(\ell \geqslant 1)$ *iff there exists a finite non-negative solution* $(y_i)_{i \in E}$ *to the inequality*

$$\begin{cases} \displaystyle\sum_{j \in E} q_{ij} y_j \leqslant -\ell \mathbb{E}_i \sigma_H^{\ell - 1}, & i \notin H, \\ \displaystyle\sum_{i \in H} \sum_{j \neq i} q_{ij} y_j < \infty. \end{cases}$$

Theorem 13 ([11, Theorem 2])**.** *Let* Q *be an irreducible regular* Q*-matrix and* H *a non-empty finite subset of* E*. Then the* Q*-process is exponentially ergodic iff* *for some $\lambda > 0$ with $\lambda < q_i$ for all $i \in E$, the following inequality has a finite non-negative solution.*

$$\begin{cases} \sum_{j\in E} q_{ij} y_j \leqslant -\lambda y_i - 1, & i \notin H, \\ \sum_{i\in H} \sum_{j\neq i} q_{ij} y_j < \infty. \end{cases}$$

Theorem 14 ([11, Theorem 2])**.** *Let Q be an irreducible regular Q-matrix and H a non-empty finite subset of E. Then the Q-process is strongly ergodic iff there exists a bounded solution $(y_i)_{i\in E}$ to the inequality*

$$\sum_{j\in E} q_{ij} y_j \leqslant -1, \qquad i \notin H.$$

3 Preliminaries for Discrete-Time Markov Chains

Let $P = (P_{ij})$ be an irreducible aperiodic transition matrix and $(X_n)_{n\geqslant 0}$ be the process. Define $\sigma_H \widehat{=} \inf\{n \geqslant 1 : X_n \in H\}$ $(H \subset E)$. Then the chain is

(1) ergodic iff $\max\limits_{i\in H} \mathbb{E}_i \sigma_H$ is finite for some (equivalently, for any) non-empty finite subset H of E;

(2) (algebraic ergodicity) ℓ-ergodic for some integer $\ell \geqslant 0$ iff $\max\limits_{i\in H} \mathbb{E}_i \sigma_H^\ell$ is finite for some (equivalently, for any) non-empty finite subset H of E;

(3) geometrically ergodic iff $\max\limits_{i\in H} \mathbb{E}_i \rho^{\sigma_H}$ is finite for some $\rho > 1$ and some (equivalently, for any) non-empty finite subset H of E;

(4) strongly ergodic iff $(\mathbb{E}_i \sigma_H)_{i\notin H}$ is bounded for some (equivalently, for any) non-empty finite subset H of E.

We have similar minimal solution assertions in time-discrete setup.

Theorem 8′ ([9, Lemma 3.1])**.** *For any $\ell \geqslant 1$, the moments of return times $\mathbb{E}_i \sigma_H, \mathbb{E}_i \sigma_H^2, \cdots, \mathbb{E}_i \sigma_H^\ell$ are inductively the minimal solution to the following ℓ-family of systems for $0 \leqslant n \leqslant \ell - 1$,*

$$x_i^{(n+1)} = \sum_{j\notin H} P_{ij} x_j^{(n+1)} + x_i^{(n)}, \qquad i \in E,$$

where $x_i^{(0)} = 1$, $i \in E$.

When $\ell = 1$, Theorem 8′ gives

Theorem 9′ ([4, Lemma 4.29])**.** $(\mathbb{E}_i\sigma_H)_{i\in E}$ *is the minimal solution to*

$$x_i = \sum_{j\notin H} P_{ij}x_j + 1, \qquad i\in E.$$

Theorem 10′. *For a non-empty finite subset H of E and $\rho > 0$, $(\mathbb{E}_i\rho^{\sigma_H})_{i\in E}$ is the minimal solution to*

$$x_i = \rho\left(\sum_{j\notin H} P_{ij}x_j + P_{iH}\right), \qquad i\in E.$$

Proof. Set

$$\begin{aligned} y_i^{(1)}(\rho) &= \rho P_{iH}, \qquad i\in E,\\ y_i^{(n+1)}(\rho) &= \rho\sum_{j\notin H} P_{ij}y_j^{(n)}(\rho), \qquad n\geqslant 1,\ i\in E. \end{aligned}$$

We claim

$$y_i^{(n)}(\rho) = \rho^n\mathbb{P}_i\left(\sigma_H = n\right), \qquad n\geqslant 1,\ i\in E.$$

In fact, $y_i^{(1)}(\rho) = \rho P_{iH} = \rho\mathbb{P}_i\left(\sigma_H = 1\right)$ for each $i\in E$. If the claim holds for some $n\geqslant 1$, then

$$\begin{aligned} y_i^{(n+1)}(\rho) &= \rho\sum_{j\notin H} P_{ij}y_j^{(n)}(\rho) \xlongequal[\text{hypothesis}]{\text{induction}} \rho\sum_{j\notin H} P_{ij}\rho^n\mathbb{P}_j\left(\sigma_H = n\right)\\ &= \rho^{n+1}\mathbb{P}_i\left(\sigma_H = n+1\right). \end{aligned}$$

So the claim is proved by induction. Because

$$\mathbb{E}_i\rho^{\sigma_H} = \sum_{n=1}^{\infty}\rho^n\mathbb{P}_i\left(\sigma_H = n\right) = \sum_{n=1}^{\infty} y_i^{(n)}(\rho),$$

$(\mathbb{E}_i\rho^{\sigma_H})_{i\in E}$ is the minimal solution by the second successive approximation scheme (cf. [4, Theorem 2.9]). □

As for time-discrete analogue of Theorems 11 to 14, we refer to [4,9] for details on these results.

Chapter 2

Main Results

In this chapter, we present our main results. Proofs will be given in Chapter 3.

1 Inverse Problems for Ergodicity of Continuous-Time Markov Chains

We begin with criteria in time-continuous case.

Theorem 15. *Let Q be an irreducible regular Q-matrix and H a non-empty finite subset of E. Then the Q-process is non-ergodic iff there is a sequence $\left\{y^{(n)}\right\}_{n=1}^{\infty}$, where $y^{(n)} = \left(y_i^{(n)}\right)_{i \in E}$ for each $n \geqslant 1$, and $\left\{y^{(n)}\right\}_{n=1}^{\infty}$ satisfies the following conditions:*

(1) $\sup\limits_{i \in E} y_i^{(n)} < \infty$ *for each* $n \geqslant 1$;

(2) *For each* $n \geqslant 1$, $\left(y_i^{(n)}\right)_{i \in E}$ *solves inequality*

$$y_i \leqslant \sum_{j \notin H} \Pi_{ij} y_j + \frac{1}{q_i}, \qquad i \in E; \tag{6}$$

(3) $\sup\limits_{n \geqslant 1} \max\limits_{i \in H} y_i^{(n)} = \infty$ *(or equivalently,* $\varlimsup\limits_{n \to \infty} \max\limits_{i \in H} y_i^{(n)} = \infty$*).*

Theorem 16. *Let Q be an irreducible regular Q-matrix and H a non-empty finite subset of E. Then the Q-process is non-strongly ergodic iff there is a sequence $\left\{y^{(n)}\right\}_{n=1}^{\infty}$, where $y^{(n)} = \left(y_i^{(n)}\right)_{i \notin H}$ for each $n \geqslant 1$, and $\left\{y^{(n)}\right\}_{n=1}^{\infty}$ satisfies the following conditions:*

(1) $\sup\limits_{i \notin H} y_i^{(n)} < \infty$ *for each* $n \geqslant 1$;

(2) *For each* $n \geqslant 1$, $\left(y_i^{(n)}\right)_{i \notin H}$ *solves inequality*

$$y_i \leqslant \sum_{j \notin H} \Pi_{ij} y_j + \frac{1}{q_i}, \qquad i \notin H; \tag{7}$$

(3) $\sup_{n \geqslant 1} \sup_{i \notin H} y_i^{(n)} = \infty$ *(or equivalently,* $\overline{\lim_{n \to \infty}} \sup_{i \notin H} y_i^{(n)} = \infty$*).*

Remark. a) We make a remark about the conditions in above two criteria. Let $\{y^{(n)}\}_{n=1}^{\infty}$ with $y^{(n)} = \left(y_i^{(n)}\right)_{i \in E}$ be a sequence satisfying the conditions in Theorem 15, then $\{\widetilde{y}^{(n)}\}_{n=1}^{\infty}$ with $\widetilde{y}^{(n)} = \left(y_i^{(n)}\right)_{i \notin H}$ is a sequence satisfying conditions in Theorem 16. We need only demonstrate condition (3). In fact, we have

$$\infty = \sup_{n \geqslant 1} \max_{i \in H} y_i^{(n)} \leqslant \sup_{n \geqslant 1} \max_{i \in H} \left(\sum_{j \notin H} \Pi_{ij} y_j^{(n)} + \frac{1}{q_i} \right) \leqslant \sup_{n \geqslant 1} \sup_{j \notin H} y_j^{(n)} + C,$$

where C is finite. And it follows that $\sup_{n \geqslant 1} \sup_{j \notin H} y_j^{(n)} = \infty$. We have illustrated the third condition in Theorem 16.

b) Testing sequence in Theorems 15 and 16 need not be non-negative. Take Theorem 15 for instance. Let $\{y^{(n)}\}_{n=1}^{\infty}$ be a sequence satisfying the conditions in Theorem 15. Then for each $n \geqslant 1$, $y^{(n)} = \left(y_i^{(n)}\right)_{i \in E}$ is a function on E. And $y^{(n)}$ is not required to be non-negative. We may even allow $\inf_{i \in E} y_i^{(n)} = -\infty$. However, $y^{(n)}$ should be a finite-valued function. In other words, for each $n \geqslant 1$ and $i \geqslant 1$, $y_i^{(n)}$ is a finite real number.

The following inverse problem criterion for algebraic ergodicity generalizes Theorem 15.

Theorem 17. *Let Q be an irreducible regular Q-matrix and H a non-empty finite subset of E. Suppose the Q-process is ℓ-ergodic for some non-negative integer ℓ, then the Q-process is not $(\ell+1)$-ergodic iff there is a sequence $\{y^{(n)}\}_{n=1}^{\infty}$, where $y^{(n)} = \left(y_i^{(n)}\right)_{i \in E}$ for each $n \geqslant 1$, and $\{y^{(n)}\}_{n=1}^{\infty}$ satisfies the following conditions:*

(1) $\sup_{i \in E} y_i^{(n)} < \infty$ *for each* $n \geqslant 1$*;*

(2) *For each* $n \geqslant 1$, $\left(y_i^{(n)}\right)_{i \in E}$ *solves inequality*

$$y_i \leqslant \sum_{j \notin H} \Pi_{ij} y_j + \frac{(\ell+1)}{q_i} \mathbb{E}_i \sigma_H^{\ell}, \qquad i \in E; \tag{8}$$

(3) $\sup_{n\geqslant 1}\max_{i\in H} y_i^{(n)}=\infty$ *(or equivalently,* $\varlimsup_{n\to\infty}\max_{i\in H} y_i^{(n)}=\infty$*).*

Theorem 18 is a non-exponential ergodicity criterion for Q-processes.

Theorem 18. *Let Q be an irreducible regular Q-matrix with $\inf_{i\in E} q_i>0$ and H a non-empty finite subset of E. Then the Q-process is non-exponentially ergodic iff there is a sequence of positive number $\{\lambda_n\}_{n=1}^{\infty}$ and a sequence of function $\{y^{(n)}\}_{n=1}^{\infty}$ on E satisfying the following conditions:*

(1) $\lim_{n\to\infty}\lambda_n=0$;

(2) *For each $n\geqslant 1$, $\left(y_i^{(n)}\right)_{i\in E}$ is finitely supported and solves inequality*

$$y_i^{(n)}\leqslant \frac{q_i}{q_i-\lambda_n}\sum_{j\notin H}\Pi_{ij}y_j^{(n)}+\frac{1}{q_i-\lambda_n},\qquad i\in E; \tag{9}$$

(3) $\sup_{n\geqslant 1}\max_{i\in H} y_i^{(n)}=\infty$ *(or equivalently,* $\varlimsup_{n\to\infty}\max_{i\in H} y_i^{(n)}=\infty$*).*

2 Inverse Problems for Ergodicity of Discrete-Time Markov Chains

Theorem 15′. *Let $P=(P_{ij})$ be an irreducible aperiodic transition matrix and H a non-empty finite subset of E. Then the chain is non-ergodic iff there is a sequence $\{y^{(n)}\}_{n=1}^{\infty}$, where $y^{(n)}=\left(y_i^{(n)}\right)_{i\in E}$ for each $n\geqslant 1$, and $\{y^{(n)}\}_{n=1}^{\infty}$ satisfies the following conditions:*

(1) $\sup_{i\in E} y_i^{(n)}<\infty$ *for each $n\geqslant 1$;*

(2) *For each $n\geqslant 1$, $\left(y_i^{(n)}\right)_{i\in E}$ solves inequality*

$$y_i\leqslant \sum_{j\notin H}P_{ij}y_j+1,\qquad i\in E; \tag{6′}$$

(3) $\sup_{n\geqslant 1}\max_{i\in H} y_i^{(n)}=\infty$ *(or equivalently,* $\varlimsup_{n\to\infty}\max_{i\in H} y_i^{(n)}=\infty$*).*

Theorem 16′. *Let $P=(P_{ij})$ be an irreducible aperiodic transition matrix and H a non-empty finite subset of E. Then the chain is non-strongly ergodic iff there is* *a sequence* $\{y^{(n)}\}_{n=1}^{\infty}$, *where* $y^{(n)} = \left(y_i^{(n)}\right)_{i\notin H}$ *for each* $n \geqslant 1$, *and* $\{y^{(n)}\}_{n=1}^{\infty}$ *satisfies the following conditions:*

(1) $\sup_{i\notin H} y_i^{(n)} < \infty$ *for each* $n \geqslant 1$;

(2) *For each* $n \geqslant 1$, $\left(y_i^{(n)}\right)_{i\notin H}$ *solves inequality*

$$y_i \leqslant \sum_{j\notin H} P_{ij} y_j + 1, \qquad i \notin H; \tag{7'}$$

(3) $\sup_{n\geqslant 1} \sup_{i\notin H} y_i^{(n)} = \infty$ *(or equivalently,* $\overline{\lim}_{n\to\infty} \sup_{i\notin H} y_i^{(n)} = \infty$*).*

Theorem 17′. *Let* $P = (P_{ij})$ *be an irreducible aperiodic transition matrix and* H *a non-empty finite subset of* E. *Suppose the chain is* ℓ*-ergodic for some non-negative integer* ℓ, *then the chain is not* $(\ell+1)$*-ergodic iff there is a sequence* $\{y^{(n)}\}_{n=1}^{\infty}$, *where* $y^{(n)} = \left(y_i^{(n)}\right)_{i\in E}$ *for each* $n \geqslant 1$, *and* $\{y^{(n)}\}_{n=1}^{\infty}$ *satisfies the following conditions:*

(1) $\sup_{i\in E} y_i^{(n)} < \infty$ *for each* $n \geqslant 1$;

(2) *For each* $n \geqslant 1$, $\left(y_i^{(n)}\right)_{i\in E}$ *solves inequality*

$$y_i \leqslant \sum_{j\notin H} P_{ij} y_j + \mathbb{E}_i \sigma_H^{\ell}, \qquad i \in E; \tag{8'}$$

(3) $\sup_{n\geqslant 1} \max_{i\in H} y_i^{(n)} = \infty$ *(or equivalently,* $\overline{\lim}_{n\to\infty} \max_{i\in H} y_i^{(n)} = \infty$*).*

Chapter 3

Proofs of Criteria for Inverse Problems

In this chapter, we assume $E = \{0, 1, 2, \cdots\}$ and $H = \{0\}$ without loss of generality. Since the proofs for discrete-time Markov Chains are similar with those for continuous-time Chains, we only give the proofs in time-continuous setup. One may easily prove time-discrete results using similar technic.

Before proceeding further, let's briefly describe the main points in our proofs. Take non-ergodicity for instance. In order to prove the expectation of return time to the state 0 is infinity, we first get a lower control for the expectation of return time. Then a sequence of increasing lower control implies the desired result. On another hand, finite approximation method would guarantee existence of an increasing sequence of lower control and therefore necessity of our conditions.

This chapter is organized as follows. Sections 1 to 4 are devoted to the proofs of Theorems 16 and 17. Then in Section 5, we prove Theorem 18.

1 Lower Control for Polynomial Moments

Theorem 19. *Let $P = (P_{ij})$ be an irreducible conservative transition matrix on E. Then the chain is transient iff the inequality*

$$\sum_{j \geqslant 0} P_{ij} z_j \leqslant z_i, \qquad i \geqslant 1$$

has a solution $z = (z_i)_{i \geqslant 0}$ satisfying

$$-\infty < \inf_{i \geqslant 0} z_i < z_0.$$

As Theorem 19 is a slight modification of [4, Theorem 4.25], its proof would not be included here.

Lemma 20. *Let ℓ be a non-negative integer and Q an irreducible regular Q-matrix on E. Assume further inequality*

$$y_i \leqslant \sum_{\substack{j\geqslant 1\\ j\neq i}} \frac{q_{ij}}{q_i} y_j + \frac{(\ell+1)}{q_i}\mathbb{E}_i\sigma_0^\ell, \qquad i \geqslant 1$$

has a finite solution $y=(y_i)_{i\geqslant 1}$ with $\sup\limits_{i\geqslant 1} y_i < \infty$. If the Q-process is $(\ell+1)$-ergodic, then we have

$$y_i \leqslant \mathbb{E}_i\sigma_0^{\ell+1}, \qquad i \geqslant 1.$$

Proof. Since the Q-process is $(\ell+1)$-ergodic, $\big(\mathbb{E}_i\sigma_0^{\ell+1}\big)_{i\geqslant 1}$ is finite and is the minimal non-negative solution to

$$x_i = \sum_{\substack{j\geqslant 1\\ j\neq i}} \frac{q_{ij}}{q_i} x_j + \frac{(\ell+1)}{q_i}\mathbb{E}_i\sigma_0^\ell, \qquad i \geqslant 1.$$

Set

$$z_i = \begin{cases} 0, & i=0,\\ \mathbb{E}_i\sigma_0^{\ell+1} - y_i, & i \geqslant 1.\end{cases}$$

Then $(z_i)_{i\geqslant 0}$ satisfies

$$\begin{cases} \displaystyle\sum_{j\geqslant 0} \Pi_{ij} z_j \leqslant z_i, \quad i \geqslant 1,\\ \displaystyle\inf_{i\geqslant 0} z_i > -\infty. \end{cases}$$

The Q-process is recurrent by our assumption, so is its embedding chain. Applying Theorem 19 to the embedding chain $\Pi = (\Pi_{ij})$, we arrive at the conclusion that

$$z_i \geqslant z_0, \qquad i \geqslant 1.$$

In other words,

$$y_i \leqslant \mathbb{E}_i\sigma_0^{\ell+1}, \qquad i \geqslant 1.$$

□

Remark 21. The hypothesis "$\sup\limits_{i\geqslant 1} y_i < \infty$" cannot be removed from Lemma 20. In fact, we consider an ergodic Q-process, then $(\mathbb{E}_i\sigma_0)_{i\geqslant 1}$ is the minimal non-negative solution to

$$x_i = \sum_{\substack{j\geqslant 1\\ j\neq i}} \frac{q_{ij}}{q_i} x_j + \frac{1}{q_i}, \qquad i \geqslant 1. \tag{10}$$

On another hand, fix an arbitrary $\varepsilon > 0$, if we take

$$x_1^{(\varepsilon)} = \mathbb{E}_1\sigma_0 + \varepsilon,$$
$$x_i^{(\varepsilon)} = x_1^{(\varepsilon)} \sum_{k=0}^{i-1} F_k^{(0)} - \sum_{k=0}^{i-1} d_k^{(0)}, \qquad i \geqslant 2,$$

then $\left(x_i^{(\varepsilon)}\right)_{i\geqslant 1}$ also solves Eq. (10) (cf. [6]). Because the Q-process is assumed to be ergodic and thus recurrent, we have $\sum_{k=0}^{\infty} F_k^{(0)} = \infty$ (cf. [4,6]). Note that

$$x_i^{(\varepsilon)} - \mathbb{E}_i\sigma_0 = \varepsilon \sum_{k=0}^{i-1} F_k^{(0)},$$

we may conclude that $\left(x_i^{(\varepsilon)}\right)_{i\geqslant 1}$ is unbounded. Meanwhile, we have

$$\mathbb{E}_i\sigma_0 < x_i^{(\varepsilon)}, \qquad i \geqslant 1.$$

This implies that the condition "$\sup_{i\geqslant 1} y_i < \infty$" cannot be removed.

It is straightforward to write the time-discrete analogue of Lemma 20 and we shall omit its proof.

Lemma 20′. *Let ℓ be a non-negative integer and P an irreducible aperiodic transition matrix on E. Assume further inequality*

$$y_i \leqslant \sum_{j\geqslant 1} P_{ij} y_j + \mathbb{E}_i\sigma_0^{\ell}, \qquad i \geqslant 1$$

has a finite solution $y = (y_i)_{i\geqslant 1}$ with $\sup_{i\geqslant 1} y_i < \infty$. If the chain is $(\ell+1)$-ergodic, then we have

$$y_i \leqslant \mathbb{E}_i\sigma_0^{\ell+1}, \qquad i \geqslant 1.$$

2 Sufficiency of Theorems 16 and 17

Proof of sufficiency of Theorem 17. If the Q-process is $(\ell+1)$-ergodic, by Theorem 8 and Lemma 20, for each $n \geqslant 1$,

$$y_0^{(n)} \leqslant \sum_{j\geqslant 1} \frac{q_{0j}}{q_0} y_j^{(n)} + \frac{(\ell+1)}{q_i}\mathbb{E}_0\sigma_0^{\ell} \leqslant \sum_{j\geqslant 1} \frac{q_{0j}}{q_0}\mathbb{E}_j\sigma_0^{\ell+1} + \frac{(\ell+1)}{q_i}\mathbb{E}_0\sigma_0^{\ell} = \mathbb{E}_0\sigma_0^{\ell+1}.$$

It follows that

$$\infty = \sup_{n\geqslant 1} y_0^{(n)} \leqslant \mathbb{E}_0\sigma_0^{\ell+1} < \infty,$$

a contradiction. □

Proof of sufficiency of Theorem 16. It suffices to prove Theorem 16 when the Q-process is ergodic. By Lemma 20 with $\ell = 0$, we have

$$y_i^{(n)} \leqslant \mathbb{E}_i \sigma_0, \qquad i \geqslant 1,\ n \geqslant 1.$$

Consequently,

$$\infty = \sup_{n\geqslant 1} \sup_{i\geqslant 1} y_i^{(n)} \leqslant \sup_{i\geqslant 1} \mathbb{E}_i \sigma_0.$$

Thus the Q-process is non-strongly ergodic. Our proof is completed. □

3 Approximation for Polynomial Moments

Let ℓ be a fixed non-negative integer. To prove necessity of Theorems 16 and 17, we consider truncated equations for each $n \geqslant 1$:

$$x_i = \sum_{\substack{1\leqslant j\leqslant n\\ j\neq i}} \frac{q_{ij}}{q_i} x_j + \frac{(\ell+1)}{q_i} \mathbb{E}_i \sigma_0^{\ell}, \qquad 1 \leqslant i \leqslant n. \tag{11.n}$$

Denote the minimal non-negative solution to Eq. (11.n) as

$$x^{(n)} = \left(x_i^{(n)},\ 1 \leqslant i \leqslant n\right).$$

Also, we set $M_n = \max_{1\leqslant i\leqslant n} x_i^{(n)}$.

Lemma 22. *If the Q-process is ℓ-erogdic, then we have*

(1) *M_n is finite for each $n \geqslant 1$;*

(2) *$\mathbb{E}_i \sigma_0^{\ell+1} = \lim_{n\to\infty}\uparrow x_i^{(n)}$ for each $i \geqslant 1$, and $(M_n)_{n\geqslant 1}$ is increasing;*

(3) *$(M_n)_{n\geqslant 1}$ is bounded iff $\left(\mathbb{E}_i \sigma_0^{\ell+1}\right)_{i\geqslant 1}$ is bounded;*

(4) *Pick $\ell = 0$, then it follows from item (3) that the Q-process is non-strongly ergodic iff $\sup_{n\geqslant 1} M_n = \infty$.*

Proof. a) Since the Q-process is ℓ-ergodic, we may pick a positive constant C_n:

$$C_n = (\ell+1) \max_{1\leqslant i\leqslant n} \mathbb{E}_i \sigma_0^{\ell} + 1.$$

Now consider inequality

$$x_i \geqslant \sum_{\substack{1\leqslant j\leqslant n\\ j\neq i}} \frac{q_{ij}}{q_i} x_j + \frac{C_n}{q_i}, \qquad 1 \leqslant i \leqslant n.$$

Introducing a change of variable $\widetilde{x}_i = \dfrac{x_i}{C_n}$, we have the following equivalent form of the above inequality:

$$\widetilde{x}_i \geqslant \sum_{\substack{1\leqslant j\leqslant n\\ j\neq i}} \frac{q_{ij}}{q_i}\widetilde{x}_j + \frac{1}{q_i}, \qquad 1\leqslant i\leqslant n. \tag{12.n}$$

By Theorem 9, the minimal solution to Eq. (12.n) is the expectation of return time to state 0 of the $Q^{(n)}$-process and is therefore finite, where $Q^{(n)}$ has the following form:

$$Q^{(n)} = \begin{pmatrix} -n & 1 & 1 & \cdots & 1 \\ q_{10} + \sum_{k=n+1}^{\infty} q_{1,k} & q_{11} & q_{12} & \cdots & q_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ q_{n0} + \sum_{k=n+1}^{\infty} q_{n,k} & q_{n1} & q_{n2} & \cdots & q_{nn} \end{pmatrix}_{(n+1)\times(n+1)}.$$

Now by Theorem 4, M_n is finite.

b) Recall Theorem 8, $\left(\mathbb{E}_i\sigma_0^{\ell+1}\right)_{i\geqslant 1}$ is the minimal solution to

$$x_i = \sum_{\substack{1\leqslant j\leqslant n\\ j\neq i}} \frac{q_{ij}}{q_i}x_j + \frac{(\ell+1)}{q_i}\mathbb{E}_i\sigma_0^{\ell}, \qquad i\geqslant 1.$$

Exploiting Theorem 5, we obtain the second assertion.

c) Some trivial manipulation leads to the other two assertions. And details are omitted. □

4 Necessity of Theorems 16 and 17

Proof of necessity of Theorem 17. Suppose the Q-process is not $(\ell+1)$-ergodic. Set

$$y_i^{(n)} = \begin{cases} \sum_{j\geqslant 1} \dfrac{q_{0j}}{q_0}x_j^{(n)} + \dfrac{(\ell+1)}{q_0}\mathbb{E}_0\sigma_0^{\ell}, & i=0, \\ x_i^{(n)}, & 1\leqslant i\leqslant n, \\ 0, & i\geqslant n+1. \end{cases}$$

By the monotone convergence theorem,

$$\begin{aligned}\lim_{n\to\infty} y_0^{(n)} &= \lim_{n\to\infty} \sum_{j\geqslant 1} \frac{q_{0j}}{q_0} y_j^{(n)} + \frac{(\ell+1)}{q_0}\mathbb{E}_0\sigma_0^{\ell} \\ &= \sum_{j\geqslant 1} \frac{q_{0j}}{q_0}\mathbb{E}_j\sigma_0^{\ell+1} + \frac{(\ell+1)}{q_0}\mathbb{E}_0\sigma_0^{\ell} \xlongequal{\textit{Theorem 8}} \mathbb{E}_0\sigma_0^{\ell+1} = \infty.\end{aligned}$$

Now it is easy to check that $\{y^{(n)}\}_{n=1}^{\infty}$ with $y^{(n)} = \left(y_i^{(n)}\right)_{i\in E}$ is a required sequence. Necessity of Theorem 17 is proved. □

Proof of necessity of Theorem 16. Assume the Q-process is non-strongly ergodic. We pick $\ell = 0$ in Lemma 22 and set

$$y_i^{(n)} = \begin{cases} x_i^{(n)}, & 1 \leqslant i \leqslant n, \\ 0, & i \geqslant n+1. \end{cases}$$

Then $\{y^{(n)}\}_{n=1}^{\infty}$ is a sequence required in Theorem 16. In fact, we may easily deduce that for each $n \geqslant 1$, $y^{(n)}$ solves Eq. (7). Meanwhile, for each $n \geqslant 1$, we have

$$\sup_{i\geqslant 1} y_i^{(n)} = M_n < \infty.$$

By the last assertion of Lemma 22, $\sup\limits_{n\geqslant 1} M_n = \infty$. Therefore,

$$\sup_{n\geqslant 1}\sup_{i\geqslant 1} y_i^{(n)} = \sup_{n\geqslant 1} M_n = \infty.$$

Hence we prove necessity of Theorem 16. □

5 Proof of Theorem 18

Since we are discussing exponential ergodicity in this section, we assume the process is ergodic without loss of generality. Our idea for proof of Theorem 18 is similar with that of Theorems 16 and 17 but technical details here are different and more complex. Briefly speaking, we first use Lemma 7 to get a lower control for exponential moment of return time. On another hand, we use finite approximation to prove the necessity.

Let Q be a Q-matrix on E with $\inf\limits_{i\in E} q_i > 0$. Fix an integer $N \geqslant 1$ and consider

Q-matrix on finite states

$$Q^{(N)} = \begin{pmatrix} -N & 1 & 1 & \cdots & 1 \\ q_{10} + \sum_{k=N+1}^{\infty} q_{1,k} & q_{11} & q_{12} & \cdots & q_{1N} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ q_{N0} + \sum_{k=N+1}^{\infty} q_{N,k} & q_{N1} & q_{N2} & \cdots & q_{NN} \end{pmatrix}_{(N+1)\times(N+1).}$$

Meanwhile, we consider the following equation for $\lambda \in \left(0, \inf_{i\in E} q_i\right)$:

$$x_i = \frac{q_i}{q_i - \lambda} \sum_{\substack{1\leqslant j\leqslant N \\ j\neq i}} \frac{q_{ij}}{q_i} x_j + \frac{1}{q_i - \lambda}, \qquad 0 \leqslant i \leqslant N. \tag{13}$$

Denote the minimal solution to Eq. (13) as $\left(x_i^{(\lambda,N)}, 0 \leqslant i \leqslant N\right)$. Then by Theorem 5, we have

$$x_0^{(\lambda,N)} \uparrow e_{00}(\lambda), \qquad \text{as } N \to \infty.$$

Also, we set $\overline{\lambda} = \dfrac{1}{2} \inf_{i\in E} q_i$.

Lemma 23. (1) *Assume the Q-process is non-exponentially ergodic, then*

$$\lim_{N\to\infty} \uparrow x_0^{\left(\overline{\lambda},N\right)} = e_{00}\left(\overline{\lambda}\right) = \infty;$$

(2) *If $x_0^{\left(\widetilde{\lambda},N\right)}$ is finite for some $\widetilde{\lambda} \in \left(0, \inf_{i\in E} q_i\right)$, then for some $\widehat{\lambda} \in \left(\widetilde{\lambda}, \inf_{i\in E} q_i\right)$, $x_0^{\left(\widehat{\lambda},N\right)}$ is finite;*

(3) *If $x_0^{\left(\widetilde{\lambda},N\right)} < \infty$ for some $\widetilde{\lambda} \in \left(0, \inf_{i\in E} q_i\right)$, then $x_0^{(\lambda,N)}$ is continuous at $\widetilde{\lambda}$ as a function of λ;*

(4) *If $x_0^{\left(\widetilde{\lambda},N\right)} = \infty$ for some $\widetilde{\lambda} \in \left(0, \inf_{i\in E} q_i\right)$, then*

$$\lim_{\lambda\uparrow\widetilde{\lambda}} x_0^{(\lambda,N)} = \infty,$$
$$x_0^{(\lambda,N)} = \infty, \qquad \lambda > \widetilde{\lambda}.$$

In other words, $x_0^{(\lambda,N)}$ is continuous at $\widetilde{\lambda}$ as an extended real-valued function;

(5) *For any fixed integer $N \geqslant 1$,*

$$\lim_{\lambda \downarrow 0} x_0^{(\lambda,N)} \leqslant \mathbb{E}_0 \sigma_0 < \infty.$$

Proof. a) The first assertion is a direct inference of Theorem 5 and non-exponential ergodicity.

b) By Eq. (13), $\left(2x_i^{(\widetilde{\lambda},N)}, 0 \leqslant i \leqslant N\right)$ is a finite solution to

$$x_i = \frac{q_i}{q_i - \widetilde{\lambda}} \sum_{\substack{1 \leqslant j \leqslant N \\ j \neq i}} \frac{q_{ij}}{q_i} x_j + \frac{2}{q_i - \widetilde{\lambda}}, \qquad 0 \leqslant i \leqslant N.$$

So it satisfies

$$x_i > \frac{q_i}{q_i - \widetilde{\lambda}} \sum_{\substack{1 \leqslant j \leqslant N \\ j \neq i}} \frac{q_{ij}}{q_i} x_j + \frac{1}{q_i - \widetilde{\lambda}}, \qquad 0 \leqslant i \leqslant N.$$

Consequently, $\left(2x_i^{(\widetilde{\lambda},N)}, 0 \leqslant i \leqslant N\right)$ also satisfies

$$x_i > \frac{q_i}{q_i - \widehat{\lambda}} \sum_{\substack{1 \leqslant j \leqslant N \\ j \neq i}} \frac{q_{ij}}{q_i} x_j + \frac{1}{q_i - \widehat{\lambda}}, \qquad 0 \leqslant i \leqslant N,$$

for some $\widehat{\lambda}$ slightly larger than $\widetilde{\lambda}$. Now by Theorem 4, $x_0^{(\widehat{\lambda},N)}$ is finite.

c) By the second conclusion, to prove the third assertion, we need only prove $x_0^{(\lambda,N)}$ is continuous on the interval $(0, \widetilde{\lambda}]$. Because

$$x_i^{(\lambda,N)} = \frac{1}{\lambda}\left(\mathbb{E}_i^{(Q^{(N)})} \mathrm{e}^{\lambda \sigma_0} - 1\right), \qquad 1 \leqslant i \leqslant N,$$

$x_i^{(\lambda,N)}$ $(i = 1, 2, \cdots, N)$ is continuous on the interval $(0, \widetilde{\lambda}]$ by Lebesgue dominated convergence theorem. Furthermore, $x_0^{(\lambda,N)}$ is continuous on the interval according to equality:

$$x_0 = \frac{q_0}{q_0 - \lambda} \sum_{1 \leqslant j \leqslant N} \frac{q_{0j}}{q_0} x_j + \frac{1}{q_0 - \lambda}.$$

d) The fourth assertion is obvious according to above discussions.

e) Now we prove the last assertion. Since the Q-process is assumed to be ergodic, $\mathbb{E}_0 \sigma_0 < \infty$. We need only illustrate

$$\lim_{\lambda \downarrow 0} x_0^{(\lambda,N)} \leqslant \mathbb{E}_0 \sigma_0.$$

By the proof of "Equivalence of Theorems 4.45 and 4.44" in [4, Page 148], we have

$$x_i^{(\lambda,N)} = \int_0^\infty \mathrm{e}^{\lambda t} \mathbb{P}_i^{\left(Q^{(N)}\right)} (\sigma_0 > t) \, \mathrm{d}t, \qquad i \geqslant 1.$$

Because the $Q^{(N)}$-process, as a process on finite state space, is exponentially ergodic, Lebesgue dominated convergence theorem gives

$$\lim_{\lambda \downarrow 0} x_i^{(\lambda,N)} = \int_0^\infty \mathbb{P}_i^{\left(Q^{(N)}\right)} (\sigma_0 > t) \, \mathrm{d}t = \mathbb{E}_i^{\left(Q^{(N)}\right)} \sigma_0 \leqslant \mathbb{E}_i \sigma_0, \qquad i \geqslant 1,$$

where the last inequality is by Theorems 5 and 9. Furthermore, by Eq. (13),

$$\begin{aligned} \lim_{\lambda \downarrow 0} x_0^{(\lambda,N)} &= \lim_{\lambda \downarrow 0} \frac{q_0}{q_0 - \lambda} \sum_{1 \leqslant j \leqslant N} \frac{q_{0j}}{q_0} x_j^{(\lambda,N)} + \lim_{\lambda \downarrow 0} \frac{1}{q_0 - \lambda} = \sum_{1 \leqslant j \leqslant N} \frac{q_{0j}}{q_0} \mathbb{E}_j^{\left(Q^{(N)}\right)} \sigma_0 + \frac{1}{q_0} \\ &\leqslant \sum_{j \geqslant 1} \frac{q_{0j}}{q_0} \mathbb{E}_j \sigma_0 + \frac{1}{q_0} = \mathbb{E}_0 \sigma_0. \end{aligned}$$

Therefore, the last assertion holds. □

Corollary 24. *For each $N \geqslant 1$, $x_0^{(\lambda,N)}$ is an extended real-valued continuous function as a funtion of λ on interval $\left(0, \overline{\lambda}\,\right]$.* □

Proof of necessity of Theorem 18. For each positive integer $n \leqslant \mathbb{E}_0 \sigma_0$, we define $y_i^{(n)} \equiv 0$ $(i \in E)$ and $\lambda_n = \overline{\lambda}$. And for each $n > \mathbb{E}_0 \sigma_0$, we now construct $y^{(n)} = \left(y_i^{(n)}\right)_{i \in E}$ and λ_n satisfying

$$y_0^{(n)} \geqslant n, \quad \lambda_n \leqslant \frac{1}{n}.$$

In fact, by the first assertion of Lemma 23, we may pick a large N_n such that

$$x_0^{\left(\overline{\lambda},N_n\right)} \geqslant n.$$

Then for each $N \geqslant N_n$,

$$x_0^{\left(\overline{\lambda},N\right)} \geqslant n.$$

Furthermore, by Corollary 24 and the last assertion of Lemma 23, for each $N \geqslant N_n$, there exists $\lambda(n,N) \in \left(0, \overline{\lambda}\,\right]$ such that

$$x_0^{(\lambda(n,N),N)} = n.$$

For ease of notation, we write $c = \inf_{N \geqslant N_n} \lambda(n,N)$. Now, we claim $c = 0$. Otherwise if $c > 0$, we have

$$e_{00}(c) = \lim_{N \to \infty} x_0^{(c,N)} \leqslant n,$$

contradicting non-exponential ergodicity.
Consequently, we may pick $\lambda\left(n, \widetilde{N_n}\right) \leqslant \frac{1}{n}$ and denote it as λ_n. Then $\lambda_n \leqslant \frac{1}{n}$ and $x_0^{\left(\lambda_n, \widetilde{N_n}\right)} = n$. Set

$$y_i^{(n)} = \begin{cases} x_i^{\left(\lambda_n, \widetilde{N_n}\right)}, & 0 \leqslant i \leqslant \widetilde{N_n}, \\ 0, & i \geqslant \widetilde{N_n} + 1. \end{cases}$$

It is now straightforward to verify that $\{\lambda_n\}_{n=1}^{\infty}$ and $\left\{y^{(n)}\right\}_{n=1}^{\infty}$ are the desired sequences. So necessity of our condition follows immediately. □

Proof of sufficiency of Theorem 18. a) We first demonstrate

$$y_0^{(n)} \leqslant e_{00}\left(\lambda_n\right), \qquad n \geqslant 1.$$

In fact, since $\left(y_i^{(n)}\right)_{i \in E}$ is finitely supported for each $n \geqslant 1$, we may pick N_n such that

$$y_i^{(n)} \leqslant \frac{q_i}{q_i - \lambda_n} \sum_{\substack{1 \leqslant j \leqslant N_n \\ j \neq i}} \frac{q_{ij}}{q_i} y_j^{(n)} + \frac{1}{q_i - \lambda_n}, \qquad 1 \leqslant i \leqslant N_n.$$

At the same time, denote the minimal solution of

$$x_i = \frac{q_i}{q_i - \lambda_n} \sum_{\substack{1 \leqslant j \leqslant N_n \\ j \neq i}} \frac{q_{ij}}{q_i} x_j + \frac{1}{q_i - \lambda_n}, \qquad 1 \leqslant i \leqslant N_n$$

as $\left(x_i^{(\lambda_n, N_n)}, 1 \leqslant i \leqslant N_n\right)$, which is positive. Then by Theorem 5 and Lemma 7,

$$y_i^{(n)} \leqslant x_i^{(\lambda_n, N_n)} \leqslant e_{i0}\left(\lambda_n\right), \qquad 1 \leqslant i \leqslant N_n.$$

It follows that

$$\begin{aligned} y_0^{(n)} &\leqslant \frac{q_0}{q_0 - \lambda_n} \sum_{1 \leqslant j \leqslant N_n} \frac{q_{0j}}{q_0} y_j^{(n)} + \frac{1}{q_0 - \lambda_n} \\ &\leqslant \frac{q_0}{q_0 - \lambda_n} \sum_{1 \leqslant j \leqslant N_n} \frac{q_{0j}}{q_0} e_{j0}\left(\lambda_n\right) + \frac{1}{q_0 - \lambda_n} \\ &\leqslant \frac{q_0}{q_0 - \lambda_n} \sum_{j \geqslant 1} \frac{q_{0j}}{q_0} e_{j0}\left(\lambda_n\right) + \frac{1}{q_0 - \lambda_n} \overset{\textit{Theorem } 10}{=\!=\!=\!=} e_{00}\left(\lambda_n\right). \end{aligned}$$

This is exactly the desired inequality.

b) For an arbitrary $\lambda > 0$, when $\lambda_n < \lambda$,

$$y_0^{(n)} \leqslant e_{00}\left(\lambda_n\right) \leqslant e_{00}(\lambda).$$

Consequently,

$$\infty = \overline{\lim_{n\to\infty}}\, y_0^{(n)} \leqslant e_{00}\left(\lambda\right).$$

It turns out that $\mathbb{E}_0 \mathrm{e}^{\lambda\sigma_0} = \infty \, (\lambda > 0)$. So the Q-process is non-exponentially ergodic. Sufficiency of Theorem 18 is proved. $\square$

Chapter 4

Some Applications

In this chapter, we shall present some applications of our criteria for inverse problems.

1 Explicit Criteria for Single Birth Processes: Alternative Proofs

Explicit and computable criteria for ergodicity and strong ergodicity of single birth processes have been studied in [13,14], respectively. In this section, we present alternative proofs (of the necessity parts) for these explicit criteria.

Let Q be an irreducible regular single birth Q-matrix on $E = \mathbb{Z}_+ = \{0, 1, 2, \cdots\}$. We have

$$q_{i,i+1} > 0,\ q_{i,i+j} = 0, \qquad i \geqslant 0,\ j \geqslant 2.$$

Define $q_n^{(k)} = \sum_{j=0}^{k} q_{nj}$ for $0 \leqslant k < n\,(k, n \geqslant 0)$ and

$$F_n^{(n)} = 1,\ F_n^{(i)} = \frac{1}{q_{n,n+1}} \sum_{k=i}^{n-1} q_n^{(k)} F_k^{(i)}, \qquad 0 \leqslant i \leqslant n,$$

$$d_0 = 0,\ d_n = \frac{1}{q_{n,n+1}} \left(1 + \sum_{k=0}^{n-1} q_n^{(k)} d_k\right) = \sum_{k=1}^{n} \frac{F_n^{(k)}}{q_{k,k+1}}, \qquad n \geqslant 1. \tag{14}$$

Also, we define

$$d = \sup_{k \geqslant 0} \frac{\sum_{n=0}^{k} d_n}{\sum_{n=0}^{k} F_n^{(0)}}.$$

It is well-known that the Q-process is recurrent iff $\sum_{n=0}^{\infty} F_n^{(0)} = \infty$ (cf. [4,6]).

Lemma 25. *Let Q be an irreducible regular single birth Q-matrix and N a positive integer. We investigate the following (truncated) equation:*

$$x_i = \sum_{\substack{1\leqslant j\leqslant N\\ j\neq i}} \frac{q_{ij}}{q_i} x_j + \frac{1}{q_i}, \qquad 1 \leqslant i \leqslant N. \tag{15}$$

(1) *Eq. (15) has a unique solution, denoted as $\left(x_1^{(N)}, x_2^{(N)}, \cdots, x_N^{(N)}\right)$;*

(2) *We have recurrence relation:*

$$x_k^{(N)} = x_1^{(N)} \sum_{n=0}^{k-1} F_n^{(0)} - \sum_{n=0}^{k-1} d_n, \qquad 1 \leqslant k \leqslant N;$$

(3) *the unique solution is positive;*

(4) $\overline{\lim\limits_{N\to\infty}}\, x_1^{(N)} \geqslant d.$

Proof. a) Eq. (15) has the following equivalent form:

$$\sum_{j=1}^{N} q_{ij} x_j = -1, \qquad 1 \leqslant i \leqslant N.$$

To prove regularity of the above linear system, we need only prove the following homogeneous equation

$$\sum_{j=1}^{N} q_{ij} x_j = 0, \qquad 1 \leqslant i \leqslant N \tag{16}$$

has only trivial solution.

Otherwise, if Eq. (16) had a non-trivial solution $(\overline{x}_1, \overline{x}_2, \cdots, \overline{x}_N)$, assume $\overline{x}_1 \geqslant 0$ without loss of generality. We claim $\overline{x}_1 \leqslant \overline{x}_2$. Since if $\overline{x}_1 > \overline{x}_2$, Eq. (16) with $i = 1$ leads to

$$0 = q_{11}\overline{x}_1 + q_{12}\overline{x}_2 < q_{11}\overline{x}_1 + q_{12}\overline{x}_1 \leqslant 0,$$

a contradiction. So we obtain $\overline{x}_1 \leqslant \overline{x}_2$. Furthermore, we may proceed to prove that $\overline{x}_k \leqslant \overline{x}_{k+1}$ using similar arguments for $k = 2, 3, \cdots, N-1$. That is

$$\overline{x}_1 \leqslant \overline{x}_2 \leqslant \cdots \leqslant \overline{x}_N.$$

Since the solution is non-trivial, we have $\overline{x}_N > 0$. Therefore,

$$\begin{aligned} 0 &= q_{N1}\overline{x}_1 + q_{N2}\overline{x}_2 + \cdots + q_{N,N-1}\overline{x}_{N-1} + q_{N,N}\overline{x}_N \\ &= (q_{N1} + q_{N2} + \cdots + q_{N,N-1} + q_{N,N})\,\overline{x}_N < 0, \end{aligned}$$

a contradiction. So Eq. (16) has only trivial solution. In this way, we prove the first assertion.

b) To prove the second assertion, we mimic the proof of [14, Lemma 2.1]. Define

$$v_0 = x_1^{(N)},\ v_n = x_{n+1}^{(N)} - x_n^{(N)}, \qquad 1 \leqslant n \leqslant N-1.$$

From Eq. (15), we easily derive that

$$v_n = \frac{1}{q_{n,n+1}} \left(\sum_{k=0}^{n-1} q_n^{(k)} v_k - 1 \right), \qquad 1 \leqslant n \leqslant N-1.$$

By induction, $v_n = v_0 F_n^{(0)} - d_n$ for $0 \leqslant n \leqslant N-1$. And our assertion follows immediately.

c) If $x_i^{(N)} = \min\limits_{1 \leqslant k \leqslant N} x_k^{(N)} \leqslant 0$, then

$$\begin{aligned} -1 = \sum_{j=1}^{N} q_{ij} x_j^{(N)} &= \sum_{j=1}^{i-1} q_{ij} \left(x_j^{(N)} - x_i^{(N)} \right) - q_{i0} x_i^{(N)} \\ &+ (1 - \delta_{i,N})\, q_{i,i+1} \left(x_{i+1}^{(N)} - x_i^{(N)} \right) - \delta_{i,N} q_{i,i+1} x_i^{(N)} \geqslant 0, \end{aligned}$$

where δ is the Kronecker delta. This contradiction infers that the unique solution is positive.

d) By the second assertion and the positiveness of the solution, we have

$$x_1^{(N)} > \max_{1 \leqslant k \leqslant N} \frac{\sum_{n=0}^{k-1} d_n}{\sum_{n=0}^{k-1} F_n^{(0)}}.$$

So the last assertion follows immediately. □

We are now in position to present our alternative proofs for explicit criteria of single birth processes.

The following ergodicity criterion is due to Shi-Jian Yan and Mu-Fa Chen [13]. Here, proof for sufficiency is picked from [13] for completeness.

Theorem 26. *Let Q be a regular single birth Q-matrix, then the Q-process is ergodic iff $d < \infty$.*

Proof. a) When $d < \infty$, we define

$$y_0 = 0,\ y_k = \sum_{n=0}^{k-1} \left(F_n^{(0)} d - d_n \right), \qquad k \geqslant 1.$$

Then $(y_i)_{i \geqslant 0}$ satisfies the condition of Theorem 11 with $H = \{0\}$. So the Q-process is ergodic when $d < \infty$.

b) When $d=\infty$, for each $N\geqslant 1$, we define

$$y_0^{(N)}=x_1^{(N)}+\frac{1}{q_1},\ y_i^{(N)}=x_i^{(N)}\,(1\leqslant i\leqslant N)\,,\ y_i^{(N)}=0\,(i\geqslant N+1)\,.$$

Because $\overline{\lim\limits_{N\to\infty}}\, x_1^{(N)}\geqslant d=\infty$, it can be easily seen that the conditions of Theorem 15 are satisfied by the sequences $\left\{y^{(N)}\right\}_{N=1}^{\infty}$ and $H=\{0\}$. So the Q-process is non-ergodic if $d=\infty$. □

The following strong ergodicity criterion is due to Yu-Hui Zhang [14].

Theorem 27. *Let Q be a regular single birth Q-matrix, then the Q-process is strongly ergodic iff* $\sup\limits_{k\geqslant 0}\sum\limits_{j=0}^{k}\left(F_j^{(0)}d-d_j\right)<\infty$.

Proof. We assume the process is ergodic without loss of generality. In light of Theorem 26, $d<\infty$ equivalently.

a) When $\sup\limits_{k\geqslant 0}\sum\limits_{j=0}^{k}\left(F_j^{(0)}d-d_j\right)<\infty$, we define

$$y_0=0,\ y_k=\sum_{n=0}^{k-1}\left(F_n^{(0)}d-d_n\right),\qquad k\geqslant 1.$$

Then $(y_i)_{i\geqslant 0}$ satisfies the condition of Theorem 14 with $H=\{0\}$. So the Q-process is strongly ergodic. This proof of sufficiency is of course not original but picked from [14].

b) When $\sup\limits_{k\geqslant 0}\sum\limits_{j=0}^{k}\left(F_j^{(0)}d-d_j\right)=\infty$, for each $N\geqslant 1$, we define

$$y_i^{(N)}=x_i^{(N)}\,(1\leqslant i\leqslant N)\,,\ y_i^{(N)}=0\,(i\geqslant N+1)\,.$$

It is obvious that $\sup\limits_{i\geqslant 1} y_i^{(N)}<\infty$ for each $N\geqslant 1$. We now show $\overline{\lim\limits_{N\to\infty}}\sup\limits_{i\geqslant 1} y_i^{(N)}=\infty$. In fact, for an arbitrary $k\geqslant 1$,

$$\begin{aligned}\overline{\lim_{N\to\infty}}\sup_{i\geqslant 1} y_i^{(N)}\geqslant \overline{\lim_{N\to\infty}}\, x_k^{(N)}&=\overline{\lim_{N\to\infty}}\sum_{n=0}^{k-1}\left(F_n^{(0)}x_1^{(N)}-d_n\right)\\&\geqslant\sum_{n=0}^{k-1}\left(F_n^{(0)}d-d_n\right).\end{aligned}$$

Taking supremum with respect to k on both sides, we obtain $\overline{\lim\limits_{N\to\infty}}\sup\limits_{i\geqslant 1} y_i^{(N)}=\infty$. The conditions of Theorem 16 are satisfied by the sequences $\left\{y^{(N)}\right\}_{N=1}^{\infty}$ and $H=\{0\}$. So the Q-process is non-strongly ergodic. □

2 A Special Class of Single Birth Processes

In this section, we study conservative single birth Q-matrix $Q = (q_{ij})$ with

$$q_{ij} = \begin{cases} i+1, & i \geqslant 0,\ j = i+1, \\ \alpha_i \geqslant 0, & i \geqslant 1,\ j = 0, \\ 0, & \text{other } i \neq j. \end{cases}$$

Assume there are infinitely many non-zero α_i, so Q is irreducible. The following illuminating example is a catalyst for this section.

Example 28. *It is obvious that the Q-process is unique for arbitrary $\{\alpha_i\}_{i=1}^{\infty}$.*

(1) *If $\alpha_i = \dfrac{1}{i^{\gamma}}$ for sufficiently large i, the Q-process is transient for $\gamma > 0$.*

(2) *If $\alpha_i = \dfrac{1}{\log^{\gamma} i}$ for sufficiently large i,*

  (a) *the Q-process is transient for $\gamma > 1$;*

  (b) *the Q-process is null recurrent for $\gamma = 1$;*

  (c) *the Q-process is ergodic but non-exponentially ergodic for $\gamma \in (0, 1)$.*

(3) *If $\alpha_i = \dfrac{1}{(\log\log i)^{\gamma}}$ for sufficiently large i, the Q-process is ergodic but non-exponentially ergodic for $\gamma > 0$.*

(4) *If*

$$\alpha_i = \begin{cases} \dfrac{1}{i}, & i \text{ is an odd positive integer}, \\ 1, & i \text{ is an even positive integer}, \end{cases}$$

the Q-process is strongly ergodic.

(5) *The Q-process is strongly ergodic if $\alpha_i \equiv 1\ \ (i \geqslant 1)$.*

Remark. It is easy to write

$$\mathbb{E}_i\sigma_0 = \sum_{k=i}^{\infty}\left(\prod_{\ell=i}^{k}\frac{\ell+1}{\ell+1+\alpha_\ell}\right)\frac{\alpha_{k+1}}{k+2+\alpha_{k+1}}\left(\sum_{\ell=i}^{k+1}\frac{1}{\ell+1+\alpha_\ell}\right)+\frac{\alpha_i}{(i+1+\alpha_i)^2}, \qquad i \geqslant 0,$$

where we put $\alpha_0 = 0$. But this explicit expression is hard to handle.

2.1 Recurrence Criterion

We begin with recurrence criterion in time-discrete case.

Lemma 29. *Let $P = (P_{ij})$ be an irreducible conservative transition matrix on $\mathbb{Z}_+ = \{0, 1, 2, \cdots\}$ with*

$$P_{ij} = \begin{cases} p_i, & i \geqslant 0,\ j = i+1, \\ 1 - p_i, & i \geqslant 0,\ j = 0, \\ 0, & \text{other } i, j \geqslant 0. \end{cases}$$

Then P is recurrent iff $\prod_{i=0}^{\infty} p_i = 0$.

Proof. By Theorems 4.24 and 4.25 in [4], we consider equation

$$(1 - p_i)\, y_0 + p_i y_{i+1} = y_i, \qquad i \geqslant 1. \tag{17}$$

Setting $y_0 = 0$, we obtain a recurrence relation:

$$y_{i+1} = \frac{1}{p_i} y_i, \qquad i \geqslant 1.$$

So Eq. (17) has a compact solution (non-constant bounded solution, respectively) if $\prod_{i=0}^{\infty} \frac{1}{p_i} = \infty$ ($< \infty$, respectively). This completes our proof. □

Corollary 30. *The Q-process is recurrent iff* $\sum_{i=1}^{\infty} \frac{\alpha_i}{i} = \infty$.

Proof. a) By Lemma 29, the Q-process is recurrent iff $\prod_{i=1}^{\infty} \frac{i+1}{i+1+\alpha_i} = 0$. Notice

$$\prod_{i=1}^{\infty} \frac{i+1}{i+1+\alpha_i} = 0 \ (\Longleftrightarrow) \ \sum_{i=1}^{\infty} \frac{\alpha_i}{i} = \infty.$$

Corollary 30 follows immediately.

b) There is also an alternative proof. By [6, Example 8.2] and using notations there, we have

$$F_0^{(0)} = 1, \quad F_i^{(0)} = \frac{\alpha_i}{i+1} \prod_{\ell=1}^{i-1} \left(1 + \frac{\alpha_\ell}{\ell+1}\right), \quad i \geqslant 1,$$

where we take the convention that $\prod_{\varnothing} = 1$.

When $\sum_{i=1}^{\infty} \frac{\alpha_i}{i} = \infty$, we have

$$\sum_{i=0}^{\infty} F_i^{(0)} \geqslant \sum_{i=1}^{\infty} \frac{\alpha_i}{i+1} \prod_{\ell=1}^{i-1} \left(1 + \frac{\alpha_\ell}{\ell+1}\right) \geqslant \sum_{i=1}^{\infty} \frac{\alpha_i}{i+1} = \infty,$$

and the Q-process is recurrent by [4, Theorem 4.52].

When $\sum_{i=1}^{\infty} \frac{\alpha_i}{i} < \infty$, we have

$$\sum_{i=1}^{\infty} F_i^{(0)} \leqslant \sum_{i=1}^{\infty} \frac{\alpha_i}{i+1} \prod_{\ell=1}^{i-1} \left(1 + \frac{\alpha_\ell}{\ell+1}\right) \leqslant C \cdot \sum_{i=1}^{\infty} \frac{\alpha_i}{i+1} < \infty.$$

Again by [4, Theorem 4.52], the Q-process is transient.

In conclusion, the Q-process is recurrent iff $\sum_{i=1}^{\infty} \frac{\alpha_i}{i} = \infty$. □

2.2 Ergodic Properties in the Case: $\lim_{i\to\infty} \alpha_i = 0$

Lemma 31. *The Q-process is non-exponentially ergodic if $\lim_{i\to\infty} \alpha_i = 0$.*

Proof. First, we deal with a special case: $\{\alpha_i\}_{i=1}^{\infty}$ is monotonically decreasing. For a fixed $n \geqslant 1$, we set

$$y_i^{(n)} = \begin{cases} \frac{1}{\alpha_i}, & 1 \leqslant i \leqslant n, \\ \frac{1}{\alpha_n}, & i \geqslant n+1. \end{cases}$$

It is straightforward to check that $y^{(n)} = \left(y_i^{(n)}\right)_{i\geqslant 1}$ satisfies

$$(i+1+\alpha_i)\, y_i^{(n)} \leqslant (i+1)\, y_{i+1}^{(n)} + 1, \qquad i \geqslant 1.$$

So $\{y^{(n)}\}_{n=1}^{\infty}$ is a sequence satisfying all conditions of Theorem 16. The Q-process is non-strongly ergodic.

Now if the Q-process is exponentially ergodic, by Theorem 10, the following Eq. (18) has a finite non-negative solution $(x_i)_{i\geqslant 1}$ for some $\lambda \in (0,1)$.

$$x_i = \frac{i+1}{i+1+\alpha_i-\lambda} x_{i+1} + \frac{1}{i+1+\alpha_i-\lambda}, \qquad i \geqslant 1. \tag{18}$$

Equivalently,

$$x_{i+1} = \frac{i+1+\alpha_i-\lambda}{i+1} x_i - \frac{1}{i+1}, \qquad i \geqslant 1.$$

Because $\lim_{i\to\infty} \alpha_i = 0$, $x_{i+1} \leqslant x_i$ for sufficiently large i. So $(x_i)_{i\geqslant 1}$ is bounded. Consequently, $\left(\frac{1}{\lambda}\left(\mathbb{E}_i e^{\lambda\sigma_0} - 1\right)\right)_{i\geqslant 1}$ is bounded since it is the minimal non-negative solution to Eq. (18). Hence $\left(\mathbb{E}_i e^{\lambda\sigma_0}\right)_{i\geqslant 1}$ is bounded and so is $(\mathbb{E}_i\sigma_0)_{i\geqslant 1}$. The Q-process is thus strongly ergodic. This is impossible. The Q-process is therefore non-exponentially ergodic.

In general case where $\{\alpha_i\}_{i=1}^{\infty}$ may not be monotonically decreasing, we define conservative $\widetilde{Q} = (\widetilde{q}_{ij})$:

$$\widetilde{q}_{ij} = \begin{cases} i+1, & i \geqslant 0,\ j = i+1, \\ \sup\limits_{k\geqslant i} \alpha_k, & i \geqslant 1,\ j = 0, \\ 0, & \text{other } i \neq j. \end{cases}$$

Because

$$\lim_{i\to\infty} \sup_{k\geqslant i} \alpha_k = \varlimsup_{i\to\infty} \alpha_i = \lim_{i\to\infty} \alpha_i = 0,$$

the $\widetilde{Q}$-process is non-exponentially ergodic according to above discussions. Consequently, the Q-process is non-exponentially ergodic by comparison. Our proof is completed. $\square$

The above proof is based on Theorem 16, we may give a more direct proof using Theorem 18.

Alternative Proof of Lemma 31. Without loss of generality, we assume $\lim_{i\to\infty}\downarrow \alpha_i = 0$. First, we set

$$\lambda_n = \frac{1}{n+1}, \qquad n \geqslant 1.$$

And for each fixed positive integer n, by Theorem 18, we consider

$$\begin{aligned} y_0^{(n)} &\leqslant \frac{1}{1-\lambda_n} y_1^{(n)} + \frac{1}{1-\lambda_n}, \\ y_i^{(n)} &\leqslant \frac{i+1}{i+1+\alpha_i-\lambda_n} y_{i+1}^{(n)} + \frac{1}{i+1+\alpha_i-\lambda_n}, \qquad i \geqslant 1. \end{aligned}$$

Introducing a change of variable $d_i^{(n)} = y_{i+1}^{(n)} - y_i^{(n)}$ $(i \geqslant 0)$, the above inequality is transformed into

$$\begin{aligned} d_0^{(n)} &\geqslant -\lambda_n y_0^{(n)} - 1, \\ d_i^{(n)} &\geqslant \frac{1}{i+1}\left(\alpha_i - \lambda_n\right) y_i^{(n)} - \frac{1}{i+1}, \qquad i \geqslant 1. \end{aligned}$$

Put $y_0^{(n)} = n$. As $\lim\limits_{i\to\infty}\!\downarrow \alpha_i = 0$, there exists M_1 such that

$$\begin{aligned} &\alpha_i \geqslant \lambda_n, && 1 \leqslant i \leqslant M_1 - 1, \\ &\alpha_i < \lambda_n, && i \geqslant M_1. \end{aligned}$$

If we place

$$\begin{aligned} d_0^{(n)} &= 0, \\ d_i^{(n)} &= \frac{\alpha_i - \lambda_n}{i+1} y_i^{(n)}, \qquad 1 \leqslant i \leqslant M_1 - 1, \end{aligned}$$

then

$$n = y_0^{(n)} = y_1^{(n)} \leqslant y_2^{(n)} \leqslant \cdots \leqslant y_{M_1}^{(n)}.$$

Furthermore, we may pick $M_2 > M_1$ such that

$$y_{M_1}^{(n)} - \frac{1}{M_1+1} - \cdots - \frac{1}{M_2} \geqslant 0,$$

$$y_{M_1}^{(n)} - \frac{1}{M_1+1} - \cdots - \frac{1}{M_2} - \frac{1}{M_2+1} < 0.$$

And we put

$$\begin{aligned} d_k^{(n)} &= -\frac{1}{k+1}, \qquad M_1 \leqslant k \leqslant M_2 - 1, \\ d_{M_2}^{(n)} &= -y_{M_2}^{(n)}, \\ d_k^{(n)} &= 0, \qquad k \geqslant M_2 + 1. \end{aligned}$$

Thus $y_k^{(n)} = 0 \, (k > M_2)$.

Now, one may check that $\{\lambda_n\}_{n=1}^{\infty}$ coupled with $\left\{y^{(n)}\right\}_{n=1}^{\infty}$ are sequences satisfying conditions in Theorem 18. The Q-process is non-exponentially ergodic. □

Lemma 32. *Suppose the Q-process is recurrent and set*

$$T = \sum_{i=1}^{\infty} \frac{1}{(i+1)\prod_{\ell=1}^{i}\left(1 + \frac{\alpha_\ell}{\ell+1}\right)}.$$

Then

(1) *the Q-process is ergodic if*

$$T < \infty \textit{ and } \varliminf_{i\to\infty} \alpha_i \prod_{\ell=1}^{i-1}\left(1 + \frac{\alpha_\ell}{\ell+1}\right) > 0;$$

(2) *the Q-process is non-ergodic if $T = \infty$.*

Proof. a) Note that

$$\sup_{i\geqslant 1}\sum_{k=1}^{i-1}\frac{1}{(k+1)\prod_{\ell=1}^{k}\left(1+\dfrac{\alpha_\ell}{\ell+1}\right)}\leqslant\sum_{k=1}^{\infty}\frac{1}{(k+1)\prod_{\ell=1}^{k}\left(1+\dfrac{\alpha_\ell}{\ell+1}\right)},$$

$$\sup_{i\geqslant 1}\frac{1}{\alpha_i\prod_{\ell=1}^{i-1}\left(1+\dfrac{\alpha_\ell}{\ell+1}\right)}\leqslant\frac{1}{\underline{\lim}_{i\to\infty}\alpha_i\prod_{\ell=1}^{i-1}\left(1+\dfrac{\alpha_\ell}{\ell+1}\right)}+C,$$

we have

$$\begin{aligned}
d&=\sup_{i\geqslant 0}\frac{\sum_{k=0}^{i}d_k}{\sum_{k=0}^{i}F_k^{(0)}}\leqslant\sup_{i\geqslant 1}\frac{d_i}{F_i^{(0)}}\overset{Eq.~(14)}{=\!=\!=}\sup_{i\geqslant 1}\frac{\sum_{k=1}^{i}\frac{F_i^{(k)}}{q_{k,k+1}}}{F_i^{(0)}}\\
&=\sup_{i\geqslant 1}\frac{\sum_{k=1}^{i-1}\dfrac{1}{k+1}\dfrac{\alpha_i}{i+1}\prod_{\ell=k+1}^{i-1}\left(1+\dfrac{\alpha_\ell}{\ell+1}\right)+\dfrac{1}{i+1}}{\dfrac{\alpha_i}{i+1}\prod_{\ell=1}^{i-1}\left(1+\dfrac{\alpha_\ell}{\ell+1}\right)}\\
&\leqslant\sup_{i\geqslant 1}\sum_{k=1}^{i-1}\frac{1}{(k+1)\prod_{\ell=1}^{k}\left(1+\dfrac{\alpha_\ell}{\ell+1}\right)}+\sup_{i\geqslant 1}\frac{1}{\alpha_i\prod_{\ell=1}^{i-1}\left(1+\dfrac{\alpha_\ell}{\ell+1}\right)},
\end{aligned}$$

where the equality in the second line is by the explicit expression of $F_i^{(k)}$ in [6, Example 8.2]. Hence $d<\infty$ under condition (1). The Q-process is ergodic by Theorem 26.

b) Under condition (2), using the O'Stolz theorem and the explicit expression of $F_i^{(k)}$ in [6, Example 8.2], we have

$$\begin{aligned}
d&=\sup_{i\geqslant 0}\frac{\sum_{k=0}^{i}d_k}{\sum_{k=0}^{i}F_k^{(0)}}\geqslant\lim_{i\to\infty}\frac{\sum_{k=0}^{i}d_k}{\sum_{k=0}^{i}F_k^{(0)}}=\lim_{i\to\infty}\frac{d_i}{F_i^{(0)}}\\
&=\lim_{i\to\infty}\frac{\sum_{k=1}^{i}\dfrac{F_i^{(k)}}{q_{k,k+1}}}{F_i^{(0)}}\geqslant\lim_{i\to\infty}\sum_{k=1}^{i-1}\frac{1}{(k+1)\prod_{\ell=1}^{k}\left(1+\dfrac{\alpha_l}{\ell+1}\right)}\\
&=\sum_{k=1}^{\infty}\frac{1}{(k+1)\prod_{\ell=1}^{k}\left(1+\dfrac{\alpha_l}{\ell+1}\right)}=\infty.
\end{aligned}$$

The Q-process is therefore non-ergodic. □

Corollary 33. *Let* $\alpha_i = \dfrac{1}{\log^{\gamma} i}$ $(i \geqslant 3)$. *Then*

(1) *the Q-process is ergodic for* $\gamma \in (0,1)$;

(2) *the Q-process is null recurrent for* $\gamma = 1$.

Proof. a) When $\gamma \in (0,1)$, we set $y_i = \log^{2\gamma} i$ $(i \geqslant 3)$. Then for sufficiently large i,

$$(i+1+\alpha_i)\, y_i \geqslant (i+1)\, y_{i+1} + 1.$$

In fact, for large i, by Lagrange mean value theorem,

$$(i+1)\left(\log^{2\gamma}(i+1) - \log^{2\gamma} i\right) \leqslant 2\gamma \frac{i+1}{i} \log^{2\gamma-1}(i+1) \leqslant \log^{\gamma} i - 1.$$

Thus, the Q-process is ergodic for $\gamma \in (0,1)$ by Theorem 11.

b) When $\gamma = 1$, by [4, Theorem 4.37], we need only prove every non-trivial non-negative solution to the following Eq. (19) is not summable.

$$\sum_{i \geqslant 0} x_i q_{ij} = 0, \qquad j \geqslant 0. \tag{19}$$

In fact, Eq. (19) gives

$$x_{i+1} = \frac{i+1}{i+2+\alpha_{i+1}} x_i, \qquad i \geqslant 0.$$

Therefore,

$$\frac{x_i}{x_{i+1}} = 1 + \frac{1}{i+1} + \frac{1}{(i+1)\log(i+1)}, \qquad i \geqslant 3.$$

Direct computation shows

$$\begin{aligned}
&\lim_{i\to\infty} \frac{\dfrac{x_i}{x_{i+1}} - 1 - \dfrac{1}{i} - \dfrac{1}{i\log i} - 0 \cdot \dfrac{1}{i \log i \log\log i}}{\dfrac{1}{i\log i \log\log i}} \\
&= \lim_{i\to\infty} \left(\frac{1}{(i+1)\log(i+1)} - \frac{1}{i\log i}\right) i \log i \log\log i \\
&= -\lim_{i\to\infty} \frac{(i+1)\log(i+1) - i\log i}{i(i+1)\log i \log(i+1)} i\log i \log\log i \\
&= -\lim_{i\to\infty} \frac{(i+1)\log(i+1) - i \log i}{(i+1)\log(i+1)} \log\log i.
\end{aligned}$$

Since

$$0 \leqslant \frac{(i+1)\log(i+1) - i\log i}{(i+1)\log(i+1)} \log\log i$$
$$\leqslant \frac{1+\log(i+1)}{(i+1)\log(i+1)} \log\log i \to 0, \qquad \text{as } i \to \infty,$$

we conclude that $(x_i)_{i\geqslant 1}$ is not summable by Kummer's test. The Q-process is thus non-ergodic. By Corollary 30, the Q-process is recurrent. Hence the Q-process is null recurrent when $\gamma = 1$.

c) There is a direct proof using Lemma 32. In fact, Kummer's test shows that $T < \infty$ for $\gamma \in (0,1)$ and $T = \infty$ for $\gamma = 1$, respectively. Besides, for $\gamma \in (0,1)$, we have

$$\alpha_i \prod_{k=1}^{i} \left(1 + \frac{\alpha_k}{k+1}\right) \geqslant \frac{1}{\log^\gamma i} \prod_{k=1}^{s} \left(1 + \frac{1}{(k+1)\log^\gamma(k+1)}\right)$$
$$\geqslant \frac{1}{\log^\gamma i} \exp\left\{\sum_{k=1}^{i} \frac{1}{2(k+1)\log(k+1)}\right\} \geqslant C\frac{1}{\log^\gamma i} \exp\left\{\int_9^i \frac{1}{x\log x}\mathrm{d}x\right\}$$
$$= C\frac{1}{\log^\gamma i}\log i = C\log^{1-\gamma} i \to \infty, \text{ as } s \to \infty.$$

Now, Corollary 33 follows immediately from Corollary 30 and Lemma 32. □

2.3 Ergodic Properties in the Case: $\varlimsup_{i\to\infty} \alpha_i > 0$

We begin with the following result for general Q-matrix.

Lemma 34. *Let Q be an irreducible regular Q-matrix and assume the Q-process is recurrent. If $\inf_{i\geqslant 1} q_{i0} > 0$, then the Q-process is strongly ergodic.*

Proof. Take $c \in (0, \inf_{i\geqslant 1} q_{i0})$, then

$$\frac{1}{c} \geqslant \frac{1}{c} + \frac{1 - \dfrac{q_{i0}}{c}}{q_i} = \frac{1}{c}\left(1 - \frac{q_{i0}}{q_i}\right) + \frac{1}{q_i} = \sum_{\substack{j\geqslant 1\\ j\neq i}} \frac{q_{ij}}{q_i}\frac{1}{c} + \frac{1}{q_i}, \qquad i \geqslant 1.$$

So the Q-process is strongly ergodic by Theorem 14. □

Lemma 35. *Suppose $\{\alpha_i\}_{i=1}^\infty$ has a subsequence $\{\alpha_{i_k}\}_{k=1}^\infty$ satisfying*

$$\inf_{k\geqslant 1} \alpha_{i_k} > 0, \quad \sup_{k\geqslant 1} \frac{i_{k+1}}{i_k} < \infty, \quad \sum_{k=1}^\infty \frac{1}{i_k} = \infty.$$

Then the Q-process is strongly ergodic.

Proof. For ease of notation, we write $i_0 = 0$. Define conservative $\widetilde{Q} = (\widetilde{q}_{ij})$:

$$\widetilde{q}_{ij} = \begin{cases} i+1, & i = i_k,\ j = i_{k+1} \text{ for some } k \geqslant 0, \\ i+1, & i_k < i < i_{k+1} \text{ for some } k \geqslant 0,\ j = i+1, \\ c \widehat{=} \dfrac{1}{2} \inf\limits_{k \geqslant 1} \alpha_{i_k}, & i = i_k \text{ for some } k \geqslant 1,\ j = 0, \\ 0, & \text{other } i \neq j. \end{cases}$$

It is easy to see that $\{i_k\}_{k=0}^{\infty}$ is an irreducible subclass of $\widetilde{Q}$. Note that $\{i_k\}_{k=0}^{\infty}$ is also a recurrent subclass of the $\widetilde{Q}$-process since $\sum\limits_{k=1}^{\infty} \dfrac{1}{i_k} = \infty$ (This is easy to illustrate using Lemma 29). Because

$$\frac{1}{c} = \frac{i_k + 1}{i_k + 1 + c} \cdot \frac{1}{c} + \frac{1}{i_k + 1 + c}, \qquad k \geqslant 1,$$

$\{i_k\}_{k=0}^{\infty}$ is furthermore a strongly ergodic subclass according to Theorem 14. Since $\sup\limits_{k \geqslant 1} \dfrac{i_{k+1}}{i_k} < \infty$ implies

$$\sup_{k \geqslant 0} \left(\frac{1}{i_k + 2} + \frac{1}{i_k + 3} + \cdots + \frac{1}{i_{k+1}} \right) \leqslant \sup_{k \geqslant 1} \frac{i_{k+1} - i_k - 1}{i_k + 1} < \infty,$$

exploiting

$$\mathbb{E}_i^{(\widetilde{Q})} \sigma_0 = \mathbb{E}_{i+1}^{(\widetilde{Q})} \sigma_0 + \frac{1}{i+1}, \qquad i_k < i < i_{k+1}, k \geqslant 0,$$

we have $\sup\limits_{i \geqslant 0} \mathbb{E}_i^{(\widetilde{Q})} \sigma_0 < \infty$. Construct an order-preserving conservative coupling Q-matrix $\overline{Q} = (\overline{q}(i, j; i', j'))$, whose marginalities are Q and $\widetilde{Q}$, with non-diagonal entries

$$\overline{q}(i, j; i', j') = \begin{cases} (i+1) \wedge (j+1), & i' = i+1,\ i \geqslant 0, \\ & j' = j+1,\ i_k < j < i_{k+1} \text{ for some } k \geqslant 0, \\ (i+1) \wedge (j+1), & i' = i+1,\ i \geqslant 0, \\ & j' = i_{k+1},\ j = i_k \text{ for some } k \geqslant 0, \\ (i-j)^+, & i' = i+1,\ i \geqslant 0,\ j' = j \geqslant 0, \\ c, & i' = 0,\ i \geqslant 1,\ j' = 0,\ j = i_k \text{ for some } k \geqslant 1, \\ \alpha_i - c, & i' = 0,\ i \geqslant 1,\ j' = j = i_k \text{ for some } k \geqslant 1, \\ \alpha_i, & i' = 0,\ i \geqslant 1,\ i_k < j' = j < i_{k+1} \text{ for some } k \geqslant 0, \\ 0, & \text{other } (i', j') \neq (i, j). \end{cases}$$

Denote the $\overline{Q}$-process as $(X(t), Y(t))_{t \geqslant 0}$, then we easily deduce that

$$\mathbb{P}_{(i_1, i_2)}^{(\overline{Q})} (X(t) \leqslant Y(t)) = 1, \qquad t > 0, i_1 \leqslant i_2.$$

Hence,

$$\sup_{i\geqslant 1}\mathbb{E}_i^{(Q)}\sigma_0 \leqslant \sup_{i\geqslant 1}\mathbb{E}_i^{(\widetilde{Q})}\sigma_0 < \infty,$$

and the Q-process is strongly ergodic. □

3 Brussel's Model and Miscellaneous Examples

Brussel's model (see [13]) is a typical model of reaction-diffusion process with several species.

Example 36. *Let S be a finite set, $E = \left(\mathbb{Z}_+^2\right)^S$ and let $p_k(u,v)$ be transition probability on S, $k=1,2$. Denote by $e_{u1}\in E$ the unit vector whose first component at site $u\in S$ is equal to 1 and the second component at u as well as other components at $v\neq u$ all equal 0. Similarly, one can define e_{u2}. The model is described by the conservative Q-matrix $Q=(q_{ij})$:*

$$q(x,y) = \begin{cases} \lambda_1 a(u), & \text{if } y = x+e_{u1},\\ \lambda_2 b(u)\,x_1(u), & \text{if } y = x-e_{u1}+e_{u2},\\ \lambda_3 \binom{x_1(u)}{2} x_2(u), & \text{if } y = x+e_{u1}-e_{u2},\\ \lambda_4 x_1(u), & \text{if } y = x-e_{u1},\\ x_k(u)\,p_k(u,v), & \text{if } y = x-e_{uk}+e_{vk},\ k=1,2,\ v\neq u,\\ 0, & \text{other } y\neq x, \end{cases}$$

and $q(x) = -q(x,x) = \sum_{y\neq x} q(x,y)$, where $x = ((x_1(u), x_2(u)) : u\in S)\in E$. a and b are positive functions on S and $\lambda_1,\cdots,\lambda_4$ are positive constants. Finite-dimensional Brussel's model is exponentially ergodic (cf. [25]). We now demonstrate that it is non-strongly ergodic, which was actually proved for the first time in [12]. But here we adopt different methods.

Proof. We shall prove our assertion by two approaches. For ease of notation, we write $\widetilde{a} = \sum_{u\in S} a(u)$, $|x| = \sum_{u\in S}(x_1(u)+x_2(u))$ for $x\in E$ and $E_i = \{x\in E : |x| = i\}$ for $i\geqslant 0$.

a) For each fixed $n\geqslant 1$, we construct function

$$F^{(n)}(x) = f_i^{(n)}, \qquad x\in E_i,\ i\geqslant 1,$$

with

$$f_i^{(n)} = \begin{cases} \dfrac{1}{\lambda_4}\log(i+1), & 1\leqslant i\leqslant n,\\ \dfrac{1}{\lambda_4}\log(n+1), & i\geqslant n+1. \end{cases}$$

Because

$$\left(1+\frac{1}{k}\right)^{\ell} \leqslant \mathrm{e}, \qquad 1 \leqslant \ell \leqslant k,$$

$$\left(1+\frac{1}{k}\right)^{\ell}\left(\frac{k+1}{k+2}\right)^{\frac{\lambda_1 \widetilde{a}}{\lambda_4}} \leqslant \mathrm{e}, \qquad 1 \leqslant \ell \leqslant k,$$

we have

$$\begin{aligned} \left(\lambda_1 \widetilde{a}+\lambda_4 \ell\right) \frac{1}{\lambda_4} \log (k+1) &\leqslant \frac{\lambda_1 \widetilde{a}}{\lambda_4} \log (k+1)+\ell \log k+1, \qquad 1 \leqslant \ell \leqslant k, \\ \left(\lambda_1 \widetilde{a}+\lambda_4 \ell\right) \frac{1}{\lambda_4} \log (k+1) &\leqslant \frac{\lambda_1 \widetilde{a}}{\lambda_4} \log (k+2)+\ell \log k+1, \qquad 1 \leqslant \ell \leqslant k. \end{aligned}$$

Now it is straightforward to check that

$$\left(\lambda_1 \widetilde{a}+\lambda_4 \sum_{u \in S} x_1(u)\right) f_i^{(n)} \leqslant \lambda_1 \widetilde{a} f_{i+1}^{(n)}+\lambda_4 \sum_{u \in S} x_1(u) f_{i-1}^{(n)}+1,$$
$$x \in E_i,\ i \geqslant 1,\ n \geqslant 1,$$

where we naturally put $f_0^{(n)}=0\,(n \geqslant 1)$.

It can be easily seen that $F^{(n)}(x)$ satisfies Eq. (7) in current setup and $\left\{F^{(n)}\right\}_{n=1}^{\infty}$ is a sequence satisfying conditions in Theorem 16. Consequently, we infer that finite-dimensional Brussel's model is non-strongly ergodic.

b) We try invoking Theorem 16 yet with a different testing sequence. For each fixed $n \geqslant 1$, we construct function

$$F^{(n)}(x)=\sum_{i=1}^{k} d_i^{(n)}, \qquad x \in E_k,\ k \geqslant 1,$$

with

$$d_i^{(n)}= \begin{cases}\dfrac{1}{\lambda_4(i+1)}, & 1 \leqslant i \leqslant n, \\ -\dfrac{1}{\lambda_1 \widetilde{a}(n+1)}, & i=n+1, \\ -\dfrac{1}{\lambda_1 \widetilde{a}}, & i \geqslant n+2.\end{cases}$$

And a trivial calculation shows that $\left\{F^{(n)}\right\}_{n=1}^{\infty}$ is a sequence satisfying conditions in Theorem 16. So finite-dimensional Brussel's model is non-strongly ergodic. □

Example 37. *Let $E = \mathbb{Z}_+^2$. Epidemic process is defined by Q-matrix $Q = (q((m,n),(m',n')) : (m,n),(m',n') \in E)$ with*

$$q((m,n),(m',n')) = \begin{cases} \alpha, & \text{if } (m',n') = (m+1,n), \\ \gamma m, & \text{if } (m',n') = (m-1,n), \\ \beta, & \text{if } (m',n') = (m,n+1), \\ \delta n, & \text{if } (m',n') = (m,n-1), \\ \varepsilon mn, & \text{if } (m',n') = (m-1,n+1), \\ 0, & \text{otherwise, unless } (m',n') = (m,n), \end{cases}$$

and $q((m,n)) = -q((m,n),(m,n)) = \sum_{(m',n')\neq(m,n)} q((m,n),(m',n'))$, where $\alpha, \gamma, \beta, \delta$, and ε are non-negative constants. We assume $\gamma > 0$ and $\delta > 0$. The Q-process is unique and ergodic when $\alpha + \beta > 0$ (cf. [1]). Epidemic process is non-strongly ergodic if $\alpha + \beta$, γ, and δ are strictly positive by [12]. Using similar argument as in Example 36, we can also carry out this result and therefore give a new proof. We will not reproduce the details here.

Example 38. *Consider a conservative birth-death Q-matrix with birth rate $b_0 = 1$, $b_i = i^\gamma$ $(i \geqslant 1)$ and death rate $a_i = i^\gamma$ $(i \geqslant 1)$. It is known that this Q-matrix is regular for all $\gamma \in \mathbb{R}$ and the Q-process is recurrent. The process is ergodic iff $\gamma > 1$ and strongly ergodic iff $\gamma > 2$ (cf. [4]). We now use Theorem 16 to demonstrate that the process is non-strongly ergodic if $\gamma \leqslant 2$. Also, we use Theorem 15 to present that the process is non-ergodic if $\gamma \leqslant 1$.*

Proof. a) First we prove the process is non-strongly ergodic if $\gamma \leqslant 2$ using Theorem 16. For each fixed $n \geqslant 1$, define

$$y_k^{(n)} = \sum_{i=1}^{k} d_i^{(n)}, \qquad k \geqslant 1,$$

with

$$d_i^{(n)} = \begin{cases} \dfrac{1}{i^{1+\frac{1}{i}}}, & 1 \leqslant i \leqslant n, \\ \dfrac{1}{i^{1+\frac{1}{n+1}}}, & i \geqslant n+1\ . \end{cases}$$

When $\gamma \leqslant 2$, we have the following estimates:

$$\frac{1}{i^{1+\frac{1}{i}}} - \frac{1}{(i+1)^{1+\frac{1}{i+1}}} \leqslant \frac{1}{i^\gamma}, \qquad i \geqslant 1, \tag{20a}$$

$$\frac{1}{i^{1+\frac{1}{n+1}}} - \frac{1}{(i+1)^{1+\frac{1}{n+1}}} \leqslant \frac{1}{i^\gamma}, \qquad i \geqslant n+1. \tag{20b}$$

In fact, Eq. (20a) holds obviously for $i = 1, 2$. Put

$$g_1(x) = \frac{1}{x^{1+\frac{1}{x}}}, \qquad x > 0.$$

Differentiating g_1, we obtain

$$|g_1'(x)| = \frac{1}{x^{2+\frac{1}{x}}}\left(1 + \frac{1-\log x}{x}\right) \leqslant \frac{1}{x^2} \leqslant \frac{1}{x^\gamma}, \qquad \text{if } x \geqslant \mathrm{e}.$$

By Lagrange mean value theorem, Eq. (20a) holds.

We turn to Eq. (20b). Denote $\varepsilon = \dfrac{1}{n+1}$, then we have

$$\begin{aligned}\frac{1}{i^{1+\varepsilon}} - \frac{1}{(i+1)^{1+\varepsilon}} &= \frac{(i+1)^{1+\varepsilon} - i^{1+\varepsilon}}{i^{1+\varepsilon}(i+1)^{1+\varepsilon}} \leqslant \frac{(1+\varepsilon)(i+1)^{\varepsilon}}{i^{1+\varepsilon}(i+1)^{1+\varepsilon}} \\ &= \frac{1+\varepsilon}{i^{1+\varepsilon}(i+1)} = \frac{1}{i^2}(1+\varepsilon)\frac{i^{1-\varepsilon}}{i+1},\end{aligned}$$

where "$\leqslant$" is obtained by mean value theorem.
Define

$$g_2(x) = (1+\varepsilon)\frac{x^{1-\varepsilon}}{x+1}, \qquad x > 0.$$

By calculus method, we see that g_2 is decreasing on the interval $[n+1, \infty)$. And one can verify easily that $g_2(n+1) \leqslant 1$. Therefore

$$g_2(i) = (1+\varepsilon)\frac{i^{1-\varepsilon}}{i+1} \leqslant 1, \qquad i \geqslant n+1.$$

And Eq. (20b) follows.
By Eq. (20), $\left(y_i^{(n)}\right)_{i\geqslant 1}$ satisfies Eq. (7) in current setup:

$$d_i^{(n)} \leqslant d_{i+1}^{(n)} + \frac{1}{i^\gamma}, \qquad i \geqslant 1. \tag{21}$$

and $\left\{y^{(n)}\right\}_{n=1}^{\infty}$ is a sequence satisfying all conditions in Theorem 16. Consequently, we conclude that the Q-process is non-strongly ergodic if $\gamma \leqslant 2$.

b) We use Theorem 16 to deduce non-strong ergodicity yet with a different testing sequence. Define

$$d_i^{(n)} = \begin{cases} \dfrac{1}{(i+9)\log(i+9)}, & 1 \leqslant i \leqslant n, \\ \dfrac{1}{(n+9)\log(n+9)} - \displaystyle\sum_{k=n}^{i-1}\frac{1}{k^2}, & i \geqslant n+1. \end{cases}$$

Because

$$\sum_{k=n}^{\infty}\frac{1}{k^2} > \int_n^{\infty}\frac{1}{x^2}\mathrm{d}x = \frac{1}{n} > \frac{1}{(n+9)\log(n+9)}, \qquad n \geqslant 1,$$

$$\frac{1}{(i+9)\log(i+9)} - \frac{1}{(i+10)\log(i+10)} \leqslant \frac{1}{i^2}, \qquad i \geqslant 1,$$

it is straightforward to verify that $\left\{y^{(n)}\right\}_{n=1}^{\infty}$, with $y_k^{(n)} = \sum_{i=1}^{k} d_i^{(n)}$ $(k \geqslant 1)$, is a sequence satisfying conditions in Theorem 16.

c) We now turn to non-ergodicity. For each $n \geqslant 1$, we set

$$y_0^{(n)} = n+1,\ y_i^{(n)} = \sum_{k=1}^{i} d_k^{(n)}, \qquad i \geqslant 1,$$

where $d_k^{(n)} = n - \sum_{j=1}^{k-1}\frac{1}{j}$ $(k \geqslant 1)$, $\sum_{\varnothing} = 0$. Hence for each $n \geqslant 1$, $\left(y_i^{(n)}\right)_{i\geqslant 0}$ satisfies

$$\begin{aligned} &y_0^{(n)} \leqslant y_1^{(n)} + 1, \\ &d_i^{(n)} \leqslant d_{i+1}^{(n)} + \frac{1}{i^{\gamma}}, \qquad i \geqslant 1, \end{aligned}$$

which is exactly Eq. (6) in current setup. And $\left\{y^{(n)}\right\}_{n=1}^{\infty}$, with $y^{(n)} = \left(y_i^{(n)}\right)_{i\geqslant 0}$, is a sequence for Theorem 15, therefore the Q-process is non-ergodic for $\gamma \leqslant 1$. □

We further investigate a multi-dimensional version of Example 38.

Example 39. *Let S be a finite set, $E = (\mathbb{Z}_+)^S$ and $p(u,v)$ a transition probability matrix on S. We denote by $\theta \in E$ whose components are identically 0 and denote by $e_u \in E$ the unit vector whose component at site $u \in S$ is equal to 1 and other components at $v \neq u$ all equal 0. Define an irreducible Q-matrix $Q = (q(x,y) : x, y \in E)$ as follows:*

$$q(x,y) = \begin{cases} x(u)^{\gamma}, & \text{if } y = x + e_u,\ x \neq \theta, \\ 1, & \text{if } x = \theta,\ y = e_u, \\ x(u)^{\gamma}, & \text{if } y = x - e_u, \\ x(u)\,p(u,v), & \text{if } y = x - e_u + e_v,\ v \neq u, \\ 0, & \text{other } y \neq x, \end{cases}$$

and $q(x) = -q(x,x) = \sum_{y\neq x} q(x,y)$, where $x = (x(u) : u \in S) \in E$. By [13, Theorem 1]*, it is easy to check that the Q-process is unique for all $\gamma \in \mathbb{R}$. We now prove*

(1) when $\gamma \leqslant 2$, the Q-process is non-strongly ergodic;

(2) when $\gamma \leqslant 1$, the Q-process is non-ergodic.

Proof. We will reduce multi-dimensional problem to 1-dimensional case. For ease of notation, we write $|x| = \sum_{u \in S} x(u)$ for $x \in E$ and $E_i = \{x \in E : |x| = i\}$ for $i \geqslant 0$.

a) Using Theorem 16, to prove that the Q-process is non-strongly ergodic for $\gamma \leqslant 2$, we need only construct sequence $\left\{F^{(n)}\right\}_{n=1}^{\infty}$ satisfying the conditions. We may guess $F^{(n)}$ is identically $f_i^{(n)}$ on E_i for each $i \geqslant 1$, and set

$$d_1^{(n)} = f_1^{(n)}, \quad d_i^{(n)} = f_i^{(n)} - f_{i-1}^{(n)} \ (i \geqslant 2).$$

And Eq. (7) now becomes

$$d_i^{(n)} \leqslant d_{i+1}^{(n)} + \frac{1}{\sum_{u \in S} x(u)^{\gamma}}, \qquad x \in E_i, \ i \geqslant 1.$$

Because

$$\sum_{u \in S} x(u)^{\gamma} \leqslant \sum_{u \in S} x(u)^2 \leqslant \left(\sum_{u \in S} x(u)\right)^2 = i^2, \qquad x \in E_i, \ i \geqslant 1, \ \gamma \leqslant 2,$$

we need only construct sequence satisfying

$$d_i^{(n)} \leqslant d_{i+1}^{(n)} + \frac{1}{i^2}, \qquad i \geqslant 1,$$

which is exactly Eq. (21) with $\gamma = 2$. Now we can proceed our proof as in Example 38. The Q-process is therefore non-strongly ergodic if $\gamma \leqslant 2$.

b) To deal with non-ergodicity, according to the discussions in a) and using similar notations, we need only consider equation

$$\begin{aligned} & y_0 \leqslant y_1 + 1, \\ & d_i \leqslant d_{i+1} + \frac{1}{i}, \qquad i \geqslant 1. \end{aligned}$$

And we can proceed as in proof c) of Example 38. Hence the multi-dimensional process is non-ergodic for $\gamma \leqslant 1$. □

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